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# Dynkin's Lemma in Measure Theory

Franz Merkl
University of Bielefeld

**Summary.** This article formalizes the proof of Dynkin's lemma in measure theory. Dynkin's lemma is a useful tool in measure theory and probability theory: it helps frequently to generalize a statement about all elements of a intersection-stable set system to all elements of the sigma-field generated by that system.

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The articles [11], [3], [13], [5], [12], [9], [14], [1], [2], [4], [10], [6], [7], and [8] provide the notation and terminology for this paper.

## 1. Preliminaries

For simplicity, we use the following convention: $O_1$, $F$ are non empty sets, $f$ is a sequence of subsets of $O_1$, $X$, $A$, $B$ are subsets of $O_1$, $D$ is a non empty subset of $2^{O_1}$, $n$, $m$ are natural numbers, and $x$, $Y$ are sets.

The following propositions are true:

(1) For every sequence $f$ of subsets of $O_1$ and for every $x$ holds $x \in \operatorname{rng} f$ iff there exists $n$ such that $f(n) = x$.

(2) For every $n$ holds $\operatorname{PSeg} n$ is finite.

Let us consider $n$. One can verify that $\operatorname{PSeg} n$ is finite.

Let $a$, $b$, $c$ be sets. The functor $a,b$ followed by $c$ is defined as follows:

(Def. 1) $a,b$ followed by $c = (\mathbb{N} \longmapsto c) {+}\!\cdot [0 \longmapsto a, 1 \longmapsto b]$.

Let $a$, $b$, $c$ be sets. One can check that $a,b$ followed by $c$ is function-like and relation-like.

Let $X$ be a non empty set and let $a$, $b$, $c$ be elements of $X$. Then $a,b$ followed by $c$ is a function from $\mathbb{N}$ into $X$.

Let $O_1$ be a non empty set and let $a$, $b$, $c$ be subsets of $O_1$. Then $a,b$ followed by $c$ is a sequence of subsets of $O_1$.

One can prove the following two propositions:

(5)[1] For all sets $a$, $b$, $c$ holds $(a,b$ followed by $c)(0) = a$ and $(a,b$ followed by $c)(1) = b$ and for every $n$ such that $n \neq 0$ and $n \neq 1$ holds $(a,b$ followed by $c)(n) = c$.

(6) For all subsets $a$, $b$ of $O_1$ holds $\bigcup \operatorname{rng}(a,b$ followed by $\emptyset) = a \cup b$.

Let $O_1$ be a non empty set, let $f$ be a sequence of subsets of $O_1$, and let $X$ be a subset of $O_1$. The functor $\operatorname{seqIntersection}(X, f)$ yields a sequence of subsets of $O_1$ and is defined as follows:

(Def. 2) For every $n$ holds $(\operatorname{seqIntersection}(X, f))(n) = X \cap f(n)$.

[1] The propositions (3) and (4) have been removed.

 

## 2. Disjoint-valued Functions and Intersection

Let us consider $O_1$ and let us consider $f$. Let us observe that $f$ is disjoint valued if and only if:

(Def. 3) If $n < m$, then $f(n)$ misses $f(m)$.

Next we state the proposition

(7) For every non empty set $Y$ and for every $x$ holds $x \subseteq \bigcap Y$ iff for every element $y$ of $Y$ holds $x \subseteq y$.

Let $x$ be a set. We introduce $x$ is intersection stable as a synonym of $x$ is multiplicative.

Let $O_1$ be a non empty set, let $f$ be a sequence of subsets of $O_1$, and let $n$ be an element of $\mathbb{N}$. The functor disjointify$(f,n)$ yielding an element of $2^{O_1}$ is defined as follows:

(Def. 5)[2] disjointify$(f,n) = f(n) \setminus \bigcup \mathrm{rng}(f \restriction \mathrm{PSeg}\, n)$.

Let $O_1$ be a non empty set and let $g$ be a sequence of subsets of $O_1$. The functor disjointify $g$ yields a sequence of subsets of $O_1$ and is defined by:

(Def. 6) For every $n$ holds (disjointify $g)(n) =$ disjointify$(g,n)$.

We now state several propositions:

(8) For every $n$ holds (disjointify $f)(n) = f(n) \setminus \bigcup \mathrm{rng}(f \restriction \mathrm{PSeg}\, n)$.

(9) For every sequence $f$ of subsets of $O_1$ holds disjointify $f$ is disjoint valued.

(10) For every sequence $f$ of subsets of $O_1$ holds $\bigcup \mathrm{rng}\,\mathrm{disjointify}\, f = \bigcup \mathrm{rng}\, f$.

(11) For all subsets $x$, $y$ of $O_1$ such that $x$ misses $y$ holds $x,y$ followed by $\emptyset_{(O_1)}$ is disjoint valued.

(12) Let $f$ be a sequence of subsets of $O_1$. Suppose $f$ is disjoint valued. Let $X$ be a subset of $O_1$. Then seqIntersection$(X,f)$ is disjoint valued.

(13) For every sequence $f$ of subsets of $O_1$ and for every subset $X$ of $O_1$ holds $X \cap \bigcup f = \bigcup$seqIntersection$(X,f)$.

## 3. Dynkin Systems: Definition and Closure Properties

Let us consider $O_1$. A subset of $2^{O_1}$ is called a Dynkin system of $O_1$ if:

(Def. 7) For every $f$ such that $\mathrm{rng}\, f \subseteq$ it and $f$ is disjoint valued holds $\bigcup f \in$ it and for every $X$ such that $X \in$ it holds $X^{\mathrm{c}} \in$ it and $\emptyset \in$ it.

Let us consider $O_1$. Note that every Dynkin system of $O_1$ is non empty.

Next we state several propositions:

(14) $2^{O_1}$ is a Dynkin system of $O_1$.

(15) If for every $Y$ such that $Y \in F$ holds $Y$ is a Dynkin system of $O_1$, then $\bigcap F$ is a Dynkin system of $O_1$.

(16) If $D$ is a Dynkin system of $O_1$ and intersection stable, then if $A \in D$ and $B \in D$, then $A \setminus B \in D$.

(17) If $D$ is a Dynkin system of $O_1$ and intersection stable, then if $A \in D$ and $B \in D$, then $A \cup B \in D$.

(18) Suppose $D$ is a Dynkin system of $O_1$ and intersection stable. Let $x$ be a finite set. If $x \subseteq D$, then $\bigcup x \in D$.

[2] The definition (Def. 4) has been removed.

(19) Suppose $D$ is a Dynkin system of $O_1$ and intersection stable. Let $f$ be a sequence of subsets of $O_1$. If $\operatorname{rng} f \subseteq D$, then $\operatorname{rng} \operatorname{disjointify} f \subseteq D$.

(20) Suppose $D$ is a Dynkin system of $O_1$ and intersection stable. Let $f$ be a sequence of subsets of $O_1$. If $\operatorname{rng} f \subseteq D$, then $\bigcup \operatorname{rng} f \in D$.

(21) For every Dynkin system $D$ of $O_1$ and for all elements $x$, $y$ of $D$ such that $x$ misses $y$ holds $x \cup y \in D$.

(22) For every Dynkin system $D$ of $O_1$ and for all elements $x$, $y$ of $D$ such that $x \subseteq y$ holds $y \setminus x \in D$.

## 4. Main Steps for Dynkin's Lemma

Next we state the proposition

(23) If $D$ is a Dynkin system of $O_1$ and intersection stable, then $D$ is a $\sigma$-field of subsets of $O_1$.

Let $O_1$ be a non empty set and let $E$ be a subset of $2^{O_1}$. The functor $\operatorname{GenDynSys} E$ yielding a Dynkin system of $O_1$ is defined as follows:

(Def. 8) $E \subseteq \operatorname{GenDynSys} E$ and for every Dynkin system $D$ of $O_1$ such that $E \subseteq D$ holds $\operatorname{GenDynSys} E \subseteq D$.

Let $O_1$ be a non empty set, let $G$ be a set, and let $X$ be a subset of $O_1$. The functor $\operatorname{DynSys}(X, G)$ yielding a subset of $2^{O_1}$ is defined by:

(Def. 9) For every subset $A$ of $O_1$ holds $A \in \operatorname{DynSys}(X, G)$ iff $A \cap X \in G$.

Let $O_1$ be a non empty set, let $G$ be a Dynkin system of $O_1$, and let $X$ be an element of $G$. Then $\operatorname{DynSys}(X, G)$ is a Dynkin system of $O_1$.

We now state four propositions:

(24) Let $E$ be a subset of $2^{O_1}$ and $X$, $Y$ be subsets of $O_1$. If $X \in E$ and $Y \in \operatorname{GenDynSys} E$ and $E$ is intersection stable, then $X \cap Y \in \operatorname{GenDynSys} E$.

(25) Let $E$ be a subset of $2^{O_1}$ and $X$, $Y$ be subsets of $O_1$. If $X \in \operatorname{GenDynSys} E$ and $Y \in \operatorname{GenDynSys} E$ and $E$ is intersection stable, then $X \cap Y \in \operatorname{GenDynSys} E$.

(26) For every subset $E$ of $2^{O_1}$ such that $E$ is intersection stable holds $\operatorname{GenDynSys} E$ is intersection stable.

(27) Let $E$ be a subset of $2^{O_1}$. Suppose $E$ is intersection stable. Let $D$ be a Dynkin system of $O_1$. If $E \subseteq D$, then $\sigma(E) \subseteq D$.

## References


[1] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[4] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_4.html`.

[5] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[6] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequence over ring and left-, right-, and bi-modules over a ring. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/algseq_1.html`.

[7] Andrzej Nędzusiak. σ-fields and probability. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/prob_1.html`.

[8] Andrzej Nędzusiak. Probability. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/prob_2.html.

[9] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[10] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[12] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[13] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[14] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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