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On Defining Functions on Trees[1]

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Piotr Rudnicki
University of Alberta
Department of Computing Science
Edmonton

Summary. The continuation of the sequence of articles on trees (see [2], [3], [4], [5]) and on context-free grammars ([13]). We define the set of complete parse trees for a given context-free grammar. Next we define the scheme of induction for the set and the scheme of defining functions by induction on the set. For each symbol of a context-free grammar we define the terminal, the pretraversal, and the posttraversal languages. The introduced terminology is tested on the example of Peano naturals.

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The articles [17], [10], [21], [19], [1], [23], [22], [8], [9], [6], [12], [14], [18], [15], [16], [7], [20], [13], [2], [3], [4], [5], and [11] provide the notation and terminology for this paper.

1. Preliminaries

The following propositions are true:

(1) For every non empty set D holds every finite sequence of elements of $\mathrm{FinTrees}(D)$ is a finite sequence of elements of $\mathrm{Trees}(D)$.

(2) For all sets x, y and for every finite sequence p of elements of x such that $y \in \mathrm{dom}\, p$ holds $p(y) \in x$.

Let X be a set. Observe that every element of X^* is relation-like and function-like.

Let X be a set. Observe that every element of X^* is finite sequence-like.

Let D be a non empty set and let t be an element of $\mathrm{FinTrees}(D)$. Note that $\mathrm{dom}\, t$ is finite.

Let D be a non empty set and let T be a set of trees decorated with elements of D. Note that every finite sequence of elements of T is decorated tree yielding.

Let D be a non empty set, let F be a non empty set of trees decorated with elements of D, and let T_1 be a non empty subset of F. We see that the element of T_1 is an element of F.

Let p be a finite sequence. Let us assume that p is decorated tree yielding. The roots of p constitute a finite sequence defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{dom}(\text{the roots of } p) = \mathrm{dom}\, p$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, p$ there exists a decorated tree T such that $T = p(i)$ and $(\text{the roots of } p)(i) = T(\emptyset)$.

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Let D be a non empty set, let T be a set of trees decorated with elements of D, and let p be a finite sequence of elements of T. Then the roots of p is a finite sequence of elements of D.

We now state four propositions:

(3) The roots of $\emptyset = \emptyset$.

(4) For every decorated tree T holds the roots of $\langle T \rangle = \langle T(\emptyset) \rangle$.

(5) Let D be a non empty set, F be a subset of FinTrees(D), and p be a finite sequence of elements of F. Suppose len (the roots of p) $= 1$. Then there exists an element x of FinTrees(D) such that $p = \langle x \rangle$ and $x \in F$.

(6) For all decorated trees T_2, T_3 holds the roots of $\langle T_2, T_3 \rangle = \langle T_2(\emptyset), T_3(\emptyset) \rangle$.

Let f be a function. The functor $\mathrm{pr1}(f)$ yields a function and is defined by:

(Def. 2) $\mathrm{dom}\,\mathrm{pr1}(f) = \mathrm{dom}\, f$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $\mathrm{pr1}(f)(x) = f(x)_{\mathbf{1}}$.

The functor $\mathrm{pr2}(f)$ yielding a function is defined as follows:

(Def. 3) $\mathrm{dom}\,\mathrm{pr2}(f) = \mathrm{dom}\, f$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $\mathrm{pr2}(f)(x) = f(x)_{\mathbf{2}}$.

Let X, Y be sets and let f be a finite sequence of elements of $[: X, Y :]$. Then $\mathrm{pr1}(f)$ is a finite sequence of elements of X. Then $\mathrm{pr2}(f)$ is a finite sequence of elements of Y.

Next we state the proposition

(7) $\mathrm{pr1}(\emptyset) = \emptyset$ and $\mathrm{pr2}(\emptyset) = \emptyset$.

The scheme *MonoSetSeq* deals with a function $\mathcal{A}$, a set $\mathcal{B}$, and a binary functor $\mathcal{F}$ yielding a set, and states that:

For all natural numbers k, s holds $\mathcal{A}(k) \subseteq \mathcal{A}(k+s)$

provided the parameters satisfy the following condition:

- For every natural number n holds $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \mathcal{F}(n, \mathcal{A}(n))$.

2. The Set of Parse Trees

Let A be a non empty set and let R be a relation between A and A^*. Note that $\langle A, R \rangle$ is non empty.

Now we present two schemes. The scheme *DTConstrStrEx* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a strict non empty tree construction structure G such that

(i) the carrier of $G = \mathcal{A}$, and

(ii) for every symbol x of G and for every finite sequence p of elements of the carrier of G holds $x \Rightarrow p$ iff $\mathcal{P}[x, p]$

for all values of the parameters.

The scheme *DTConstrStrUniq* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

Let G_1, G_2 be strict non empty tree construction structures. Suppose that

(i) the carrier of $G_1 = \mathcal{A}$,

(ii) for every symbol x of G_1 and for every finite sequence p of elements of the carrier of G_1 holds $x \Rightarrow p$ iff $\mathcal{P}[x, p]$,

(iii) the carrier of $G_2 = \mathcal{A}$, and

(iv) for every symbol x of G_2 and for every finite sequence p of elements of the carrier of G_2 holds $x \Rightarrow p$ iff $\mathcal{P}[x, p]$.

Then $G_1 = G_2$

for all values of the parameters.

We now state the proposition

(8) For every non empty tree construction structure G holds the terminals of G misses the nonterminals of G.

Now we present four schemes. The scheme *DTCMin* deals with a function $\mathcal{A}$, a non empty tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

There exists a subset X of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]) such that

(i) $X = \bigcup \mathcal{A}$,

(ii) for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $\langle d, \mathcal{F}(d) \rangle \in X$,

(iii) for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of X such that $o \Rightarrow$ pr1(the roots of p) holds $\langle o, \mathcal{G}(o, \mathrm{pr1}(\text{the roots of } p), \mathrm{pr2}(\text{the roots of } p)) \rangle$-tree$(p) \in X$, and

(iv) for every subset F of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]) such that for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $\langle d, \mathcal{F}(d) \rangle \in F$ and for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of F such that $o \Rightarrow$ pr1(the roots of p) holds $\langle o, \mathcal{G}(o, \mathrm{pr1}(\text{the roots of } p), \mathrm{pr2}(\text{the roots of } p)) \rangle$-tree$(p) \in F$ holds $X \subseteq F$

provided the parameters meet the following conditions:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d \rangle$; t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$: $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \emptyset \wedge d = \mathcal{G}(t, \emptyset, \emptyset)\}$, and
- Let n be a natural number. Then $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \{\langle o, \mathcal{G}(o, \mathrm{pr1}(\text{the roots of } p), \mathrm{pr2}(\text{the roots of } p)) \rangle$-tree$(p)$; o ranges over symbols of $\mathcal{B}$, p ranges over elements of $\mathcal{A}(n)^*$: $\bigvee_{q : \text{finite sequence of elements of FinTrees([: the carrier of } \mathcal{B}, \mathcal{C} :])} (p = q \wedge o \Rightarrow \mathrm{pr1}(\text{the roots of } q))\}$.

The scheme *DTCSymbols* deals with a function $\mathcal{A}$, a non empty tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

There exists a subset X_1 of FinTrees(the carrier of $\mathcal{B}$) such that

(i) $X_1 = \{t_{\mathbf{1}}$; t ranges over elements of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]): $t \in \bigcup \mathcal{A}\}$,

(ii) for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $d \in X_1$,

(iii) for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of X_1 such that $o \Rightarrow$ the roots of p holds o-tree$(p) \in X_1$, and

(iv) for every subset F of FinTrees(the carrier of $\mathcal{B}$) such that for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $d \in F$ and for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of F such that $o \Rightarrow$ the roots of p holds o-tree$(p) \in F$ holds $X_1 \subseteq F$

provided the parameters meet the following conditions:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d \rangle$; t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$: $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \emptyset \wedge d = \mathcal{G}(t, \emptyset, \emptyset)\}$, and
- Let n be a natural number. Then $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \{\langle o, \mathcal{G}(o, \mathrm{pr1}(\text{the roots of } p), \mathrm{pr2}(\text{the roots of } p)) \rangle$-tree$(p)$; o ranges over symbols of $\mathcal{B}$, p ranges over elements of $\mathcal{A}(n)^*$: $\bigvee_{q : \text{finite sequence of elements of FinTrees([: the carrier of } \mathcal{B}, \mathcal{C} :])} (p = q \wedge o \Rightarrow \mathrm{pr1}(\text{the roots of } q))\}$.

The scheme *DTCHeight* deals with a function $\mathcal{A}$, a non empty tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

Let n be a natural number and d_1 be an element of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]). If $d_1 \in \bigcup \mathcal{A}$, then $d_1 \in \mathcal{A}(n)$ iff $\operatorname{height} \operatorname{dom} d_1 \leq n$

provided the parameters meet the following requirements:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d \rangle$; t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$: $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \emptyset \wedge d = \mathcal{G}(t, \emptyset, \emptyset)\}$, and
- Let n be a natural number. Then $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \{\langle o, \mathcal{G}(o, \mathrm{pr1}(\text{the roots of } p), \mathrm{pr2}(\text{the roots of } p)) \rangle$-tree$(p)$; o ranges over symbols of $\mathcal{B}$, p ranges over elements

of $\mathcal{A}(n)^*$: $\bigvee_{q\,:\,\text{finite sequence of elements of FinTrees}([:\text{the carrier of } \mathcal{B},\, \mathcal{C}:])}$ $(p = q \wedge o \Rightarrow \mathrm{pr1}($the roots of $q))\}$.

The scheme *DTCUniq* deals with a function $\mathcal{A}$, a non empty tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

Let d_2, d_3 be trees decorated with elements of $[:$the carrier of $\mathcal{B}$, $\mathcal{C}:]$. If $d_2 \in \bigcup \mathcal{A}$ and $d_3 \in \bigcup \mathcal{A}$ and $(d_2)_{\mathbf{1}} = (d_3)_{\mathbf{1}}$, then $d_2 = d_3$

provided the following conditions are satisfied:

- $\operatorname{dom}\mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d\rangle$; t ranges over symbols of $\mathcal{B}, d$ ranges over elements of $\mathcal{C}: t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \emptyset \wedge d = \mathcal{G}(t, \emptyset, \emptyset)\}$, and
- Let n be a natural number. Then $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \{\langle o,\, \mathcal{G}(o, \mathrm{pr1}($the roots of $p), \mathrm{pr2}($the roots of $p))\rangle$-tree(p); o ranges over symbols of $\mathcal{B}$, p ranges over elements of $\mathcal{A}(n)^*$: $\bigvee_{q\,:\,\text{finite sequence of elements of FinTrees}([:\text{the carrier of } \mathcal{B},\, \mathcal{C}:])}$ $(p = q \wedge o \Rightarrow \mathrm{pr1}($the roots of $q))\}$.

Let G be a non empty tree construction structure. The functor $\mathrm{TS}(G)$ yields a subset of FinTrees(the carrier of G) and is defined by the conditions (Def. 4).

(Def. 4)(i) For every symbol d of G such that $d \in$ the terminals of G holds the root tree of $d \in \mathrm{TS}(G)$,

(ii) for every symbol o of G and for every finite sequence p of elements of $\mathrm{TS}(G)$ such that $o \Rightarrow$ the roots of p holds o-tree$(p) \in \mathrm{TS}(G)$, and

(iii) for every subset F of FinTrees(the carrier of G) such that for every symbol d of G such that $d \in$ the terminals of G holds the root tree of $d \in F$ and for every symbol o of G and for every finite sequence p of elements of F such that $o \Rightarrow$ the roots of p holds o-tree$(p) \in F$ holds $\mathrm{TS}(G) \subseteq F$.

Now we present three schemes. The scheme *DTConstrInd* deals with a non empty tree construction structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every tree t decorated with elements of the carrier of $\mathcal{A}$ such that $t \in \mathrm{TS}(\mathcal{A})$ holds $\mathcal{P}[t]$

provided the following conditions are satisfied:

- For every symbol s of $\mathcal{A}$ such that $s \in$ the terminals of $\mathcal{A}$ holds $\mathcal{P}[$the root tree of $s]$, and
- Let n_1 be a symbol of $\mathcal{A}$ and t_1 be a finite sequence of elements of $\mathrm{TS}(\mathcal{A})$. Suppose that
  (i) $n_1 \Rightarrow$ the roots of t_1, and
  (ii) for every tree t decorated with elements of the carrier of $\mathcal{A}$ such that $t \in \operatorname{rng} t_1$ holds $\mathcal{P}[t]$.

  Then $\mathcal{P}[n_1$-tree$(t_1)]$.

The scheme *DTConstrIndDef* deals with a non empty tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

There exists a function f from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$ such that

(i) for every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $f($the root tree of $t) = \mathcal{F}(t)$, and

(ii) for every symbol n_1 of $\mathcal{A}$ and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathcal{A})$ such that $n_1 \Rightarrow$ the roots of t_1 holds $f(n_1$-tree$(t_1)) = \mathcal{G}(n_1,$the roots of t_1, $f \cdot t_1)$

for all values of the parameters.

The scheme *DTConstrUniqDef* deals with a non empty tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and functions $\mathcal{C}$, $\mathcal{D}$ from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the following conditions are met:

- (i) For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{C}($the root tree of $t) = \mathcal{F}(t)$, and

(ii) for every symbol n_1 of $\mathcal{A}$ and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathcal{A})$ such that $n_1 \Rightarrow$ the roots of t_1 holds $\mathcal{C}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, \text{the roots of } t_1, \mathcal{C} \cdot t_1)$, and

- (i) For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{D}$(the root tree of t) $= \mathcal{F}(t)$, and
  (ii) for every symbol n_1 of $\mathcal{A}$ and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathcal{A})$ such that $n_1 \Rightarrow$ the roots of t_1 holds $\mathcal{D}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, \text{the roots of } t_1, \mathcal{D} \cdot t_1)$.

3. An example: Peano naturals

The strict non empty tree construction structure $\mathbb{N}_{\text{Peano}}$ is defined by the conditions (Def. 5).

(Def. 5)(i) The carrier of $\mathbb{N}_{\text{Peano}} = \{0, 1\}$, and

(ii) for every symbol x of $\mathbb{N}_{\text{Peano}}$ and for every finite sequence y of elements of the carrier of $\mathbb{N}_{\text{Peano}}$ holds $x \Rightarrow y$ iff $x = 1$ but $y = \langle 0 \rangle$ or $y = \langle 1 \rangle$.

4. Properties of parse trees

Let G be a non empty tree construction structure. We say that G has terminals if and only if:

(Def. 6) The terminals of $G \neq \emptyset$.

We say that G has nonterminals if and only if:

(Def. 7) The nonterminals of $G \neq \emptyset$.

We say that G has useful nonterminals if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let n_1 be a symbol of G. Suppose $n_1 \in$ the nonterminals of G. Then there exists a finite sequence p of elements of $\mathrm{TS}(G)$ such that $n_1 \Rightarrow$ the roots of p.

Let us note that there exists a non empty tree construction structure which is strict and has terminals, nonterminals, and useful nonterminals.

Let G be a non empty tree construction structure with terminals. Then the terminals of G is a non empty subset of G. Observe that $\mathrm{TS}(G)$ is non empty.

Let G be a non empty tree construction structure with useful nonterminals. Note that $\mathrm{TS}(G)$ is non empty.

Let G be a non empty tree construction structure with nonterminals. Then the nonterminals of G is a non empty subset of G.

Let G be a non empty tree construction structure with terminals. A terminal of G is an element of the terminals of G.

Let G be a non empty tree construction structure with nonterminals. A nonterminal of G is an element of the nonterminals of G.

Let G be a non empty tree construction structure with nonterminals and useful nonterminals and let n_1 be a nonterminal of G. A finite sequence of elements of $\mathrm{TS}(G)$ is said to be a subtree sequence joinable by n_1 if:

(Def. 9) $n_1 \Rightarrow$ the roots of it.

Let G be a non empty tree construction structure with terminals and let t be a terminal of G. Then the root tree of t is an element of $\mathrm{TS}(G)$.

Let G be a non empty tree construction structure with nonterminals and useful nonterminals, let n_1 be a nonterminal of G, and let p be a subtree sequence joinable by n_1. Then n_1-tree(p) is an element of $\mathrm{TS}(G)$.

We now state two propositions:

(9) Let G be a non empty tree construction structure with terminals, t_2 be an element of $\mathrm{TS}(G)$, and s be a terminal of G. If $t_2(\emptyset) = s$, then $t_2 =$ the root tree of s.

(10) Let G be a non empty tree construction structure with terminals and nonterminals, t_2 be an element of $\mathrm{TS}(G)$, and n_1 be a nonterminal of G. Suppose $t_2(\emptyset) = n_1$. Then there exists a finite sequence t_1 of elements of $\mathrm{TS}(G)$ such that $t_2 = n_1$-tree(t_1) and $n_1 \Rightarrow$ the roots of t_1.

5. THE EXAMPLE CONTINUED

Let us note that $\mathbb{N}_{\rm Peano}$ has terminals, nonterminals, and useful nonterminals.

Let n_1 be a nonterminal of $\mathbb{N}_{\rm Peano}$ and let t be an element of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$. Then n_1-tree(t) is an element of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$.

Let x be a finite sequence of elements of $\mathbb{N}$. Let us assume that $x \neq \emptyset$. The functor $(x)({}_1{+}1)$ yielding a natural number is defined as follows:

(Def. 10) There exists a natural number n such that $(x)({}_1{+}1) = n+1$ and $x(1) = n$.

The function $\mathbb{N}_{\rm Peano} \to \mathbb{N}$ from $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ into $\mathbb{N}$ is defined by the conditions (Def. 11).

(Def. 11)(i) For every symbol t of $\mathbb{N}_{\rm Peano}$ such that $t \in$ the terminals of $\mathbb{N}_{\rm Peano}$ holds $(\mathbb{N}_{\rm Peano} \to \mathbb{N})$(the root tree of t) $= 0$, and

(ii) for every symbol n_1 of $\mathbb{N}_{\rm Peano}$ and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ such that $n_1 \Rightarrow$ the roots of t_1 holds $(\mathbb{N}_{\rm Peano} \to \mathbb{N})(n_1\text{-tree}(t_1)) = ((\mathbb{N}_{\rm Peano} \to \mathbb{N}) \cdot t_1)({}_1{+}1)$.

Let x be an element of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$. The functor $\mathrm{succ}(x)$ yields an element of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ and is defined by:

(Def. 12) $\mathrm{succ}(x) = 1\text{-tree}(\langle x \rangle)$.

The function $\mathbb{N} \to \mathbb{N}_{\rm Peano}$ from $\mathbb{N}$ into $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ is defined as follows:

(Def. 13) $(\mathbb{N} \to \mathbb{N}_{\rm Peano})(0) =$ the root tree of 0 and for every natural number n holds $(\mathbb{N} \to \mathbb{N}_{\rm Peano})(n+1) = \mathrm{succ}((\mathbb{N} \to \mathbb{N}_{\rm Peano})(n))$.

The following propositions are true:

(11) For every element p_1 of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ holds $p_1 = (\mathbb{N} \to \mathbb{N}_{\rm Peano})((\mathbb{N}_{\rm Peano} \to \mathbb{N})(p_1))$.

(12) For every natural number n holds $n = (\mathbb{N}_{\rm Peano} \to \mathbb{N})((\mathbb{N} \to \mathbb{N}_{\rm Peano})(n))$.

6. TREE TRAVERSALS AND TERMINAL LANGUAGE

Let D be a set and let F be a finite sequence of elements of D^*. The functor $\mathrm{Flat}(F)$ yielding an element of D^* is defined as follows:

(Def. 14) There exists a binary operation g on D^* such that for all elements p, q of D^* holds $g(p, q) = p \frown q$ and $\mathrm{Flat}(F) = g \odot F$.

One can prove the following proposition

(13) For every set D and for every element d of D^* holds $\mathrm{Flat}(\langle d \rangle) = d$.

Let G be a non empty tree construction structure and let t_2 be a tree decorated with elements of the carrier of G. Let us assume that $t_2 \in \mathrm{TS}(G)$. The terminals of t_2 constitute a finite sequence of elements of the terminals of G defined by the condition (Def. 15).

(Def. 15) There exists a function f from $\mathrm{TS}(G)$ into (the terminals of G)* such that

(i) the terminals of $t_2 = f(t_2)$,

(ii) for every symbol t of G such that $t \in$ the terminals of G holds f(the root tree of t) $= \langle t \rangle$, and

(iii) for every symbol n_1 of G and for every finite sequence t_1 of elements of $\mathrm{TS}(G)$ such that $n_1 \Rightarrow$ the roots of t_1 holds $f(n_1\text{-tree}(t_1)) = \mathrm{Flat}(f \cdot t_1)$.

The pretraversal string of t_2 is a finite sequence of elements of the carrier of G and is defined by the condition (Def. 16).

(Def. 16) There exists a function f from $\mathrm{TS}(G)$ into (the carrier of $G)^*$ such that

(i) the pretraversal string of $t_2 = f(t_2)$,

(ii) for every symbol t of G such that $t \in$ the terminals of G holds f(the root tree of $t) = \langle t \rangle$, and

(iii) for every symbol n_1 of G and for every finite sequence t_1 of elements of $\mathrm{TS}(G)$ and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of (the carrier of $G)^*$ such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \langle n_1 \rangle \frown \mathrm{Flat}(x)$.

The posttraversal string of t_2 is a finite sequence of elements of the carrier of G and is defined by the condition (Def. 17).

(Def. 17) There exists a function f from $\mathrm{TS}(G)$ into (the carrier of $G)^*$ such that

(i) the posttraversal string of $t_2 = f(t_2)$,

(ii) for every symbol t of G such that $t \in$ the terminals of G holds f(the root tree of $t) = \langle t \rangle$, and

(iii) for every symbol n_1 of G and for every finite sequence t_1 of elements of $\mathrm{TS}(G)$ and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of (the carrier of $G)^*$ such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \mathrm{Flat}(x) \frown \langle n_1 \rangle$.

Let G be a non empty non empty tree construction structure with nonterminals and let n_1 be a symbol of G. The language derivable from n_1 is a subset of (the terminals of $G)^*$ and is defined by the condition (Def. 18).

(Def. 18) The language derivable from $n_1 = \{$the terminals of t_2; t_2 ranges over elements of FinTrees(the carrier of G): $t_2 \in \mathrm{TS}(G) \wedge t_2(\emptyset) = n_1\}$.

The language of pretraversals derivable from n_1 is a subset of (the carrier of $G)^*$ and is defined by the condition (Def. 19).

(Def. 19) The language of pretraversals derivable from $n_1 = \{$the pretraversal string of t_2; t_2 ranges over elements of FinTrees(the carrier of G): $t_2 \in \mathrm{TS}(G) \wedge t_2(\emptyset) = n_1\}$.

The language of posttraversals derivable from n_1 is a subset of (the carrier of $G)^*$ and is defined by the condition (Def. 20).

(Def. 20) The language of posttraversals derivable from $n_1 = \{$the posttraversal string of t_2; t_2 ranges over elements of FinTrees(the carrier of G): $t_2 \in \mathrm{TS}(G) \wedge t_2(\emptyset) = n_1\}$.

Next we state several propositions:

(14) For every tree t decorated with elements of the carrier of $\mathbb{N}_{\mathrm{Peano}}$ such that $t \in \mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the terminals of $t = \langle 0 \rangle$.

(15) For every symbol n_1 of $\mathbb{N}_{\mathrm{Peano}}$ holds the language derivable from $n_1 = \{\langle 0 \rangle\}$.

(16) For every element t of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the pretraversal string of $t = (\mathrm{height}\,\mathrm{dom}\,t \mapsto 1) \frown \langle 0 \rangle$.

(17) Let n_1 be a symbol of $\mathbb{N}_{\mathrm{Peano}}$. Then

(i) if $n_1 = 0$, then the language of pretraversals derivable from $n_1 = \{\langle 0 \rangle\}$, and

(ii) if $n_1 = 1$, then the language of pretraversals derivable from $n_1 = \{(n \mapsto 1) \frown \langle 0 \rangle; n$ ranges over natural numbers: $n \neq 0\}$.

(18) For every element t of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the posttraversal string of $t = \langle 0 \rangle \frown (\mathrm{height}\,\mathrm{dom}\,t \mapsto 1)$.

(19) Let n_1 be a symbol of $\mathbb{N}_{\mathrm{Peano}}$. Then

(i) if $n_1 = 0$, then the language of posttraversals derivable from $n_1 = \{\langle 0 \rangle\}$, and

(ii) if $n_1 = 1$, then the language of posttraversals derivable from $n_1 = \{\langle 0 \rangle \frown (n \mapsto 1); n$ ranges over natural numbers: $n \neq 0\}$.

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