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Dickson's Lemma

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Summary. We present a Mizar formalization of the proof of Dickson's lemma following [7], chapters 4.2 and 4.3.

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The articles [29], [11], [35], [22], [36], [38], [28], [37], [15], [30], [3], [33], [34], [8], [26], [27], [5], [32], [2], [1], [24], [16], [17], [10], [9], [20], [25], [19], [14], [31], [23], [4], [18], [12], [6], [13], and [21] provide the notation and terminology for this paper.

1. PRELIMINARIES

The following two propositions are true:

(1) For every function g and for every set x such that $\mathrm{dom}\, g = \{x\}$ holds $g = x \mathrel{\dot{\longmapsto}} g(x)$.

(2) For every natural number n holds $n \subseteq n+1$.

The scheme *FinSegRng2* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(i); i$ ranges over natural numbers: $\mathcal{A} < i \wedge i \leq \mathcal{B} \wedge \mathcal{P}[i]\}$ is finite

for all values of the parameters.

One can prove the following proposition

(3) For every infinite set X holds there exists a function from $\mathbb{N}$ into X which is one-to-one.

Let R be a relational structure and let f be a sequence of R. We say that f is ascending if and only if:

(Def. 1) For every natural number n holds $f(n+1) \neq f(n)$ and $\langle f(n), f(n+1) \rangle \in$ the internal relation of R.

Let R be a relational structure and let f be a sequence of R. We say that f is weakly ascending if and only if:

(Def. 2) For every natural number n holds $\langle f(n), f(n+1) \rangle \in$ the internal relation of R.

One can prove the following four propositions:

(4) Let R be a non empty transitive relational structure and f be a sequence of R. Suppose f is weakly ascending. Let i, j be natural numbers. If $i < j$, then $f(i) \leq f(j)$.

(5) Let R be a non empty relational structure. Then R is connected if and only if the internal relation of R is strongly connected in the carrier of R.

(7)[1] Let L be a relational structure, Y be a set, and a be a set. Then (the internal relation of L)-Seg(a) misses Y and $a \in Y$ if and only if a is minimal w.r.t. Y, the internal relation of L.

(8) Let L be a non empty transitive antisymmetric relational structure, x be an element of L, and a, N be sets. Suppose a is minimal w.r.t. (the internal relation of L)-Seg$(x) \cap N$, the internal relation of L. Then a is minimal w.r.t. N, the internal relation of L.

2. More on Ordering Relations

Let R be a relational structure. We say that R is quasi ordered if and only if:

(Def. 3) R is reflexive and transitive.

Let R be a relational structure. Let us assume that R is quasi ordered. The functor $\mathrm{EqRel}(R)$ yields an equivalence relation of the carrier of R and is defined as follows:

(Def. 4) $\mathrm{EqRel}(R) = (\text{the internal relation of } R) \cap (\text{the internal relation of } R)^{\smile}$.

Next we state the proposition

(9) Let R be a relational structure and x, y be elements of R. If R is quasi ordered, then $x \in [y]_{\mathrm{EqRel}(R)}$ iff $x \leq y$ and $y \leq x$.

Let R be a relational structure. The functor $\leq_E R$ yielding a binary relation on $\mathrm{Classes}\,\mathrm{EqRel}(R)$ is defined by:

(Def. 5) For all sets A, B holds $\langle A, B \rangle \in \leq_E R$ iff there exist elements a, b of R such that $A = [a]_{\mathrm{EqRel}(R)}$ and $B = [b]_{\mathrm{EqRel}(R)}$ and $a \leq b$.

Next we state two propositions:

(10) For every relational structure R such that R is quasi ordered holds $\leq_E R$ partially orders $\mathrm{Classes}\,\mathrm{EqRel}(R)$.

(11) Let R be a non empty relational structure. If R is quasi ordered and connected, then $\leq_E R$ linearly orders $\mathrm{Classes}\,\mathrm{EqRel}(R)$.

Let R be a binary relation. The functor $R\backslash^{\smile}$ yields a binary relation and is defined as follows:

(Def. 6) $R\backslash^{\smile} = R \setminus R^{\smile}$.

Let R be a binary relation. Observe that $R\backslash^{\smile}$ is asymmetric.

Let X be a set and let R be a binary relation on X. Then $R\backslash^{\smile}$ is a binary relation on X.

Let R be a relational structure. The functor $R\backslash^{\smile}$ yielding a strict relational structure is defined by:

(Def. 7) $R\backslash^{\smile} = \langle \text{the carrier of } R, \text{the internal relation of } R\backslash^{\smile} \rangle$.

Let R be a non empty relational structure. One can verify that $R\backslash^{\smile}$ is non empty.

Let R be a transitive relational structure. Observe that $R\backslash^{\smile}$ is transitive.

Let R be a relational structure. Note that $R\backslash^{\smile}$ is antisymmetric.

We now state several propositions:

(12) For every non empty poset R and for every element x of R holds $[x]_{\mathrm{EqRel}(R)} = \{x\}$.

(13) For every binary relation R holds $R = R\backslash^{\smile}$ iff R is asymmetric.

[1] The proposition (6) has been removed.

(14) For every binary relation R such that R is transitive holds $R\backslash^{\smile}$ is transitive.

(15) Let R be a binary relation and a, b be sets. If R is antisymmetric, then $\langle a, b\rangle \in R\backslash^{\smile}$ iff $\langle a, b\rangle \in R$ and $a \neq b$.

(16) For every relational structure R such that R is well founded holds $R\backslash^{\smile}$ is well founded.

(17) For every relational structure R such that $R\backslash^{\smile}$ is well founded and R is antisymmetric holds R is well founded.

3. Foundedness Properties

One can prove the following propositions:

(18) Let L be a relational structure, N be a set, and x be an element of $L\backslash^{\smile}$. Then x is minimal w.r.t. N, the internal relation of $L\backslash^{\smile}$ if and only if $x \in N$ and for every element y of L such that $y \in N$ and $\langle y, x\rangle \in$ the internal relation of L holds $\langle x, y\rangle \in$ the internal relation of L.

(19) Let R, S be non empty relational structures and m be a map from R into S. Suppose that

(i) R is quasi ordered,

(ii) S is antisymmetric,

(iii) $S\backslash^{\smile}$ is well founded, and

(iv) for all elements a, b of R holds if $a \leq b$, then $m(a) \leq m(b)$ and if $m(a) = m(b)$, then $\langle a, b\rangle \in \mathrm{EqRel}(R)$.

Then $R\backslash^{\smile}$ is well founded.

Let R be a non empty relational structure and let N be a subset of R. The functor MinClasses N yielding a family of subsets of R is defined by the condition (Def. 8).

(Def. 8) Let x be a set. Then $x \in \mathrm{MinClasses}\,N$ if and only if there exists an element y of $R\backslash^{\smile}$ such that y is minimal w.r.t. N, the internal relation of $R\backslash^{\smile}$ and $x = [y]_{\mathrm{EqRel}(R)} \cap N$.

We now state several propositions:

(20) Let R be a non empty relational structure, N be a subset of R, and x be a set. Suppose R is quasi ordered and $x \in \mathrm{MinClasses}\,N$. Let y be an element of $R\backslash^{\smile}$. If $y \in x$, then y is minimal w.r.t. N, the internal relation of $R\backslash^{\smile}$.

(21) Let R be a non empty relational structure. Then $R\backslash^{\smile}$ is well founded if and only if for every subset N of R such that $N \neq \emptyset$ there exists a set x such that $x \in \mathrm{MinClasses}\,N$.

(22) Let R be a non empty relational structure, N be a subset of R, and y be an element of $R\backslash^{\smile}$. If y is minimal w.r.t. N, the internal relation of $R\backslash^{\smile}$, then MinClasses N is non empty.

(23) Let R be a non empty relational structure, N be a subset of R, and x be a set. If R is quasi ordered and $x \in \mathrm{MinClasses}\,N$, then x is non empty.

(24) Let R be a non empty relational structure. Suppose R is quasi ordered. Then R is connected and $R\backslash^{\smile}$ is well founded if and only if for every non empty subset N of R holds $\overline{\overline{\mathrm{MinClasses}\,N}} = 1$.

(25) Let R be a non empty poset. Then the internal relation of R well orders the carrier of R if and only if for every non empty subset N of R holds $\overline{\overline{\mathrm{MinClasses}\,N}} = 1$.

Let R be a relational structure, let N be a subset of R, and let B be a set. We say that B is Dickson basis of N, R if and only if:

(Def. 9) $B \subseteq N$ and for every element a of R such that $a \in N$ there exists an element b of R such that $b \in B$ and $b \leq a$.

We now state two propositions:

(26) For every relational structure R holds $\emptyset$ is Dickson basis of $\emptyset_{\text{the carrier of } R}$, R.

(27) Let R be a non empty relational structure, N be a non empty subset of R, and B be a set. If B is Dickson basis of N, R, then B is non empty.

Let R be a relational structure. We say that R is Dickson if and only if:

(Def. 10) For every subset N of R holds there exists a set which is Dickson basis of N, R and finite.

Next we state two propositions:

(28) For every non empty relational structure R such that $R\backslash^{\smile}$ is well founded and R is connected holds R is Dickson.

(29) Let R, S be relational structures. Suppose that

(i) the internal relation of $R \subseteq$ the internal relation of S,

(ii) R is Dickson, and

(iii) the carrier of $R =$ the carrier of S.

Then S is Dickson.

Let f be a function and let b be a set. Let us assume that $\operatorname{dom} f = \mathbb{N}$ and $b \in \operatorname{rng} f$. The functor $f \operatorname{mindex} b$ yields a natural number and is defined as follows:

(Def. 11) $f(f \operatorname{mindex} b) = b$ and for every natural number i such that $f(i) = b$ holds $f \operatorname{mindex} b \leq i$.

Let R be a non empty 1-sorted structure, let f be a sequence of R, let b be a set, and let m be a natural number. Let us assume that there exists a natural number j such that $m < j$ and $f(j) = b$. The functor $f \operatorname{mindex}(b,m)$ yielding a natural number is defined by:

(Def. 12) $f(f \operatorname{mindex}(b,m)) = b$ and $m < f \operatorname{mindex}(b,m)$ and for every natural number i such that $m < i$ and $f(i) = b$ holds $f \operatorname{mindex}(b,m) \leq i$.

We now state several propositions:

(30) Let R be a non empty relational structure. Suppose R is quasi ordered and Dickson. Let f be a sequence of R. Then there exist natural numbers i, j such that $i < j$ and $f(i) \leq f(j)$.

(31) Let R be a relational structure, N be a subset of R, and x be an element of $R\backslash^{\smile}$. Suppose R is quasi ordered and $x \in N$ and (the internal relation of R)-Seg$(x) \cap N \subseteq [x]_{\mathrm{EqRel}(R)}$. Then x is minimal w.r.t. N, the internal relation of $R\backslash^{\smile}$.

(32) Let R be a non empty relational structure. Suppose R is quasi ordered and for every sequence f of R there exist natural numbers i, j such that $i < j$ and $f(i) \leq f(j)$. Let N be a non empty subset of R. Then MinClasses N is finite and MinClasses N is non empty.

(33) Let R be a non empty relational structure. Suppose R is quasi ordered and for every non empty subset N of R holds MinClasses N is finite and MinClasses N is non empty. Then R is Dickson.

(34) For every non empty relational structure R such that R is quasi ordered and Dickson holds $R\backslash^{\smile}$ is well founded.

(35) Let R be a non empty poset and N be a non empty subset of R. Suppose R is Dickson. Then there exists a set B such that B is Dickson basis of N, R and for every set C such that C is Dickson basis of N, R holds $B \subseteq C$.

Let R be a non empty relational structure and let N be a subset of R. Let us assume that R is Dickson. The functor Dickson-Bases(N,R) yielding a non empty family of subsets of R is defined by:

(Def. 13) For every set B holds $B \in$ Dickson-Bases(N,R) iff B is Dickson basis of N, R.

Next we state several propositions:

(36) Let R be a non empty relational structure and s be a sequence of R. If R is Dickson, then there exists a sequence of R which is a subsequence of s and weakly ascending.

(37) For every relational structure R such that R is empty holds R is Dickson.

(38) Let M, N be relational structures. Suppose M is Dickson and N is Dickson and M is quasi ordered and N is quasi ordered. Then $[:M, N:]$ is quasi ordered and $[:M, N:]$ is Dickson.

(39) Let R, S be relational structures. Suppose R and S are isomorphic and R is Dickson and quasi ordered. Then S is quasi ordered and Dickson.

(40) Let p be a relational structure yielding many sorted set indexed by 1 and z be an element of 1. Then $p(z)$ and $\prod p$ are isomorphic.

Let X be a set, let p be a relational structure yielding many sorted set indexed by X, and let Y be a subset of X. Note that $p{\restriction}Y$ is relational structure yielding.

We now state three propositions:

(41) Let n be a non empty natural number and p be a relational structure yielding many sorted set indexed by n. Then $\prod p$ is non empty if and only if p is nonempty.

(42) Let n be a non empty natural number, p be a relational structure yielding many sorted set indexed by $n+1$, n_1 be a subset of $n+1$, and n_2 be an element of $n+1$. If $n_1 = n$ and $n_2 = n$, then $[:\prod(p{\restriction}n_1),\ p(n_2):]$ and $\prod p$ are isomorphic.

(43) Let n be a non empty natural number and p be a relational structure yielding many sorted set indexed by n. Suppose that for every element i of n holds $p(i)$ is Dickson and $p(i)$ is quasi ordered. Then $\prod p$ is quasi ordered and $\prod p$ is Dickson.

Let p be a relational structure yielding many sorted set indexed by $\emptyset$. One can check the following observations:

* $\prod p$ is non empty,
* $\prod p$ is antisymmetric,
* $\prod p$ is quasi ordered, and
* $\prod p$ is Dickson.

The binary relation NATOrd on $\mathbb{N}$ is defined as follows:

(Def. 14) NATOrd $= \{\langle x, y\rangle; x$ ranges over elements of $\mathbb{N}$, y ranges over elements of $\mathbb{N}$: $x \leq y\}$.

The following four propositions are true:

(44) NATOrd is reflexive in $\mathbb{N}$.

(45) NATOrd is antisymmetric in $\mathbb{N}$.

(46) NATOrd is strongly connected in $\mathbb{N}$.

(47) NATOrd is transitive in $\mathbb{N}$.

The non empty relational structure OrderedNAT is defined by:

(Def. 15) OrderedNAT $= \langle \mathbb{N}, \text{NATOrd}\rangle$.

One can check the following observations:

* OrderedNAT is connected,
* OrderedNAT is Dickson,
* OrderedNAT is quasi ordered,
* OrderedNAT is antisymmetric,
* OrderedNAT is transitive, and
* OrderedNAT is well founded.

Let n be a natural number. One can check the following observations:

* $\prod(n \longmapsto \text{OrderedNAT})$ is non empty,
* $\prod(n \longmapsto \text{OrderedNAT})$ is Dickson,
* $\prod(n \longmapsto \text{OrderedNAT})$ is quasi ordered, and
* $\prod(n \longmapsto \text{OrderedNAT})$ is antisymmetric.

One can prove the following propositions:

(48) Let M be a relational structure. Suppose M is Dickson and quasi ordered. Then $[:M, \text{OrderedNAT}:]$ is quasi ordered and $[:M, \text{OrderedNAT}:]$ is Dickson.

(49) Let R, S be non empty relational structures. Suppose that

(i) R is Dickson and quasi ordered,

(ii) S is quasi ordered,

(iii) the internal relation of $R \subseteq$ the internal relation of S, and

(iv) the carrier of $R =$ the carrier of S.

Then $S\backslash^{\smile}$ is well founded.

(50) Let R be a non empty relational structure. Suppose R is quasi ordered. Then R is Dickson if and only if for every non empty relational structure S such that S is quasi ordered and the internal relation of $R \subseteq$ the internal relation of S and the carrier of $R =$ the carrier of S holds $S\backslash^{\smile}$ is well founded.

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