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Calculus of Quantifiers. Deduction Theorem

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Summary. Some tautologies of the Classical Quantifier Calculus. The deduction theorem is also proved.

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The articles [9], [4], [11], [2], [3], [10], [8], [7], [1], [5], and [6] provide the notation and terminology for this paper.

For simplicity, we use the following convention: X denotes a subset of CQC-WFF, F, G, p, q, r denote elements of CQC-WFF, s, h denote formulae, and x, y denote bound variables.

Next we state a number of propositions:

(1) If $\vdash p \Rightarrow (q \Rightarrow r)$, then $\vdash p \wedge q \Rightarrow r$.

(2) If $\vdash p \Rightarrow (q \Rightarrow r)$, then $\vdash q \wedge p \Rightarrow r$.

(3) If $\vdash p \wedge q \Rightarrow r$, then $\vdash p \Rightarrow (q \Rightarrow r)$.

(4) If $\vdash p \wedge q \Rightarrow r$, then $\vdash q \Rightarrow (p \Rightarrow r)$.

(5) $y \in \mathrm{snb}(\forall_x s)$ iff $y \in \mathrm{snb}(s)$ and $y \neq x$.

(6) $y \in \mathrm{snb}(\exists_x s)$ iff $y \in \mathrm{snb}(s)$ and $y \neq x$.

(7) $y \in \mathrm{snb}(s \Rightarrow h)$ iff $y \in \mathrm{snb}(s)$ or $y \in \mathrm{snb}(h)$.

(9)[1] $y \in \mathrm{snb}(s \wedge h)$ iff $y \in \mathrm{snb}(s)$ or $y \in \mathrm{snb}(h)$.

(10) $y \in \mathrm{snb}(s \vee h)$ iff $y \in \mathrm{snb}(s)$ or $y \in \mathrm{snb}(h)$.

(11) $x \notin \mathrm{snb}(\forall_{x,y} s)$ and $y \notin \mathrm{snb}(\forall_{x,y} s)$.

(12) $x \notin \mathrm{snb}(\exists_{x,y} s)$ and $y \notin \mathrm{snb}(\exists_{x,y} s)$.

(14)[2] $(s \Rightarrow h)(x) = s(x) \Rightarrow h(x)$.

(15) $(s \vee h)(x) = s(x) \vee h(x)$.

(17)[3] If $x \neq y$, then $(\exists_x p)(y) = \exists_x p(y)$.

(18) $\vdash p \Rightarrow \exists_x p$.

[1] The proposition (8) has been removed.
[2] The proposition (13) has been removed.
[3] The proposition (16) has been removed.

(19) If $\vdash p$, then $\vdash \exists_x p$.

(20) $\vdash \forall_x p \Rightarrow \exists_x p$.

(21) $\vdash \forall_x p \Rightarrow \exists_y p$.

(22) If $\vdash p \Rightarrow q$ and $x \notin \mathrm{snb}(q)$, then $\vdash \exists_x p \Rightarrow q$.

(23) If $x \notin \mathrm{snb}(p)$, then $\vdash \exists_x p \Rightarrow p$.

(24) If $x \notin \mathrm{snb}(p)$ and $\vdash \exists_x p$, then $\vdash p$.

(25) If $p = h(x)$ and $q = h(y)$ and $y \notin \mathrm{snb}(h)$, then $\vdash p \Rightarrow \exists_y q$.

(26) If $\vdash p$, then $\vdash \forall_x p$.

(27) If $x \notin \mathrm{snb}(p)$, then $\vdash p \Rightarrow \forall_x p$.

(28) If $p = h(x)$ and $q = h(y)$ and $x \notin \mathrm{snb}(h)$, then $\vdash \forall_x p \Rightarrow q$.

(29) If $y \notin \mathrm{snb}(p)$, then $\vdash \forall_x p \Rightarrow \forall_y p$.

(30) If $p = h(x)$ and $q = h(y)$ and $x \notin \mathrm{snb}(h)$ and $y \notin \mathrm{snb}(p)$, then $\vdash \forall_x p \Rightarrow \forall_y q$.

(31) If $x \notin \mathrm{snb}(p)$, then $\vdash \exists_x p \Rightarrow \exists_y p$.

(32) If $p = h(x)$ and $q = h(y)$ and $x \notin \mathrm{snb}(q)$ and $y \notin \mathrm{snb}(h)$, then $\vdash \exists_x p \Rightarrow \exists_y q$.

(34)[4] $\vdash \forall_x (p \Rightarrow q) \Rightarrow (\forall_x p \Rightarrow \forall_x q)$.

(35) If $\vdash \forall_x (p \Rightarrow q)$, then $\vdash \forall_x p \Rightarrow \forall_x q$.

(36) $\vdash \forall_x (p \Leftrightarrow q) \Rightarrow (\forall_x p \Leftrightarrow \forall_x q)$.

(37) If $\vdash \forall_x (p \Leftrightarrow q)$, then $\vdash \forall_x p \Leftrightarrow \forall_x q$.

(38) $\vdash \forall_x (p \Rightarrow q) \Rightarrow (\exists_x p \Rightarrow \exists_x q)$.

(39) If $\vdash \forall_x (p \Rightarrow q)$, then $\vdash \exists_x p \Rightarrow \exists_x q$.

(40) $\vdash \forall_x (p \wedge q) \Rightarrow \forall_x p \wedge \forall_x q$ and $\vdash \forall_x p \wedge \forall_x q \Rightarrow \forall_x (p \wedge q)$.

(41) $\vdash \forall_x (p \wedge q) \Leftrightarrow \forall_x p \wedge \forall_x q$.

(42) $\vdash \forall_x (p \wedge q)$ iff $\vdash \forall_x p \wedge \forall_x q$.

(43) $\vdash \forall_x p \vee \forall_x q \Rightarrow \forall_x (p \vee q)$.

(44) $\vdash \exists_x (p \vee q) \Rightarrow \exists_x p \vee \exists_x q$ and $\vdash \exists_x p \vee \exists_x q \Rightarrow \exists_x (p \vee q)$.

(45) $\vdash \exists_x (p \vee q) \Leftrightarrow \exists_x p \vee \exists_x q$.

(46) $\vdash \exists_x (p \vee q)$ iff $\vdash \exists_x p \vee \exists_x q$.

(47) $\vdash \exists_x (p \wedge q) \Rightarrow \exists_x p \wedge \exists_x q$.

(48) If $\vdash \exists_x (p \wedge q)$, then $\vdash \exists_x p \wedge \exists_x q$.

(49) $\vdash \forall_x \neg\neg p \Rightarrow \forall_x p$ and $\vdash \forall_x p \Rightarrow \forall_x \neg\neg p$.

(50) $\vdash \forall_x \neg\neg p \Leftrightarrow \forall_x p$.

(51) $\vdash \exists_x \neg\neg p \Rightarrow \exists_x p$ and $\vdash \exists_x p \Rightarrow \exists_x \neg\neg p$.

(52) $\vdash \exists_x \neg\neg p \Leftrightarrow \exists_x p$.

[4] The proposition (33) has been removed.

(53) $\vdash \neg\exists_x\neg p \Rightarrow \forall_x p$ and $\vdash \forall_x p \Rightarrow \neg\exists_x\neg p$.

(54) $\vdash \neg\exists_x\neg p \Leftrightarrow \forall_x p$.

(55) $\vdash \neg\forall_x p \Rightarrow \exists_x\neg p$ and $\vdash \exists_x\neg p \Rightarrow \neg\forall_x p$.

(56) $\vdash \neg\forall_x p \Leftrightarrow \exists_x\neg p$.

(57) $\vdash \neg\exists_x p \Rightarrow \forall_x\neg p$ and $\vdash \forall_x\neg p \Rightarrow \neg\exists_x p$.

(58) $\vdash \forall_x\neg p \Leftrightarrow \neg\exists_x p$.

(59) $\vdash \forall_x\forall_y p \Rightarrow \forall_y\forall_x p$ and $\vdash \forall_{x,y} p \Rightarrow \forall_{y,x} p$.

(60) If $p = h(x)$ and $q = h(y)$ and $y \notin \mathrm{snb}(h)$, then $\vdash \forall_x\forall_y q \Rightarrow \forall_x p$.

(61) $\vdash \exists_x\exists_y p \Rightarrow \exists_y\exists_x p$ and $\vdash \exists_{x,y} p \Rightarrow \exists_{y,x} p$.

(62) If $p = h(x)$ and $q = h(y)$ and $y \notin \mathrm{snb}(h)$, then $\vdash \exists_x p \Rightarrow \exists_{x,y} q$.

(63) $\vdash \exists_x\forall_y p \Rightarrow \forall_y\exists_x p$.

(64) $\vdash \exists_x(p \Leftrightarrow p)$.

(65) $\vdash \exists_x(p \Rightarrow q) \Rightarrow (\forall_x p \Rightarrow \exists_x q)$ and $\vdash (\forall_x p \Rightarrow \exists_x q) \Rightarrow \exists_x(p \Rightarrow q)$.

(66) $\vdash \exists_x(p \Rightarrow q) \Leftrightarrow (\forall_x p \Rightarrow \exists_x q)$.

(67) $\vdash \exists_x(p \Rightarrow q)$ iff $\vdash \forall_x p \Rightarrow \exists_x q$.

(68) $\vdash \forall_x(p \wedge q) \Rightarrow p \wedge \forall_x q$.

(69) $\vdash \forall_x(p \wedge q) \Rightarrow \forall_x p \wedge q$.

(70) If $x \notin \mathrm{snb}(p)$, then $\vdash p \wedge \forall_x q \Rightarrow \forall_x(p \wedge q)$.

(71) If $x \notin \mathrm{snb}(p)$ and $\vdash p \wedge \forall_x q$, then $\vdash \forall_x(p \wedge q)$.

(72) If $x \notin \mathrm{snb}(p)$, then $\vdash p \vee \forall_x q \Rightarrow \forall_x(p \vee q)$ and $\vdash \forall_x(p \vee q) \Rightarrow p \vee \forall_x q$.

(73) If $x \notin \mathrm{snb}(p)$, then $\vdash p \vee \forall_x q \Leftrightarrow \forall_x(p \vee q)$.

(74) If $x \notin \mathrm{snb}(p)$, then $\vdash p \vee \forall_x q$ iff $\vdash \forall_x(p \vee q)$.

(75) If $x \notin \mathrm{snb}(p)$, then $\vdash p \wedge \exists_x q \Rightarrow \exists_x(p \wedge q)$ and $\vdash \exists_x(p \wedge q) \Rightarrow p \wedge \exists_x q$.

(76) If $x \notin \mathrm{snb}(p)$, then $\vdash p \wedge \exists_x q \Leftrightarrow \exists_x(p \wedge q)$.

(77) If $x \notin \mathrm{snb}(p)$, then $\vdash p \wedge \exists_x q$ iff $\vdash \exists_x(p \wedge q)$.

(78) If $x \notin \mathrm{snb}(p)$, then $\vdash \forall_x(p \Rightarrow q) \Rightarrow (p \Rightarrow \forall_x q)$ and $\vdash (p \Rightarrow \forall_x q) \Rightarrow \forall_x(p \Rightarrow q)$.

(79) If $x \notin \mathrm{snb}(p)$, then $\vdash (p \Rightarrow \forall_x q) \Leftrightarrow \forall_x(p \Rightarrow q)$.

(80) If $x \notin \mathrm{snb}(p)$, then $\vdash \forall_x(p \Rightarrow q)$ iff $\vdash p \Rightarrow \forall_x q$.

(81) If $x \notin \mathrm{snb}(q)$, then $\vdash \exists_x(p \Rightarrow q) \Rightarrow (\forall_x p \Rightarrow q)$.

(82) $\vdash (\forall_x p \Rightarrow q) \Rightarrow \exists_x(p \Rightarrow q)$.

(83) If $x \notin \mathrm{snb}(q)$, then $\vdash \forall_x p \Rightarrow q$ iff $\vdash \exists_x(p \Rightarrow q)$.

(84) If $x \notin \mathrm{snb}(q)$, then $\vdash (\exists_x p \Rightarrow q) \Rightarrow \forall_x(p \Rightarrow q)$ and $\vdash \forall_x(p \Rightarrow q) \Rightarrow (\exists_x p \Rightarrow q)$.

(85) If $x \notin \mathrm{snb}(q)$, then $\vdash (\exists_x p \Rightarrow q) \Leftrightarrow \forall_x(p \Rightarrow q)$.

(86) If $x \notin \mathrm{snb}(q)$, then $\vdash \exists_x p \Rightarrow q$ iff $\vdash \forall_x(p \Rightarrow q)$.

(87) If $x \notin \mathrm{snb}(p)$, then $\vdash \exists_x (p \Rightarrow q) \Rightarrow (p \Rightarrow \exists_x q)$.

(88) $\vdash (p \Rightarrow \exists_x q) \Rightarrow \exists_x (p \Rightarrow q)$.

(89) If $x \notin \mathrm{snb}(p)$, then $\vdash (p \Rightarrow \exists_x q) \Leftrightarrow \exists_x (p \Rightarrow q)$.

(90) If $x \notin \mathrm{snb}(p)$, then $\vdash p \Rightarrow \exists_x q$ iff $\vdash \exists_x (p \Rightarrow q)$.

(91) $\{p\} \vdash p$.

(92) $\mathrm{Cn}(\{p\} \cup \{q\}) = \mathrm{Cn}\{p \wedge q\}$.

(93) $\{p, q\} \vdash r$ iff $\{p \wedge q\} \vdash r$.

(94) If $X \vdash p$, then $X \vdash \forall_x p$.

(95) If $x \notin \mathrm{snb}(p)$, then $X \vdash \forall_x (p \Rightarrow q) \Rightarrow (p \Rightarrow \forall_x q)$.

(96) If F is closed and $X \cup \{F\} \vdash G$, then $X \vdash F \Rightarrow G$.

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