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Convex Sets and Convex Combinations

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Summary. Convexity is one of the most important concepts in a study of analysis. Especially, it has been applied around the optimization problem widely. Our purpose is to define the concept of convexity of a set on Mizar, and to develop the generalities of convex analysis. The construction of this article is as follows: Convexity of the set is defined in the section 1. The section 2 gives the definition of convex combination which is a kind of the linear combination and related theorems are proved there. In section 3, we define the convex hull which is an intersection of all convex sets including a given set. The last section is some theorems which are necessary to compose this article.

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The articles [12], [11], [17], [9], [13], [3], [1], [8], [4], [2], [15], [14], [16], [5], [10], [6], and [7] provide the notation and terminology for this paper.

1. Convex Sets

Let V be a non empty RLS structure, let M be a subset of V, and let r be a real number. The functor $r \cdot M$ yielding a subset of V is defined by:

(Def. 1) $r \cdot M = \{r \cdot v; v$ ranges over elements of $V\colon v \in M\}$.

Let V be a non empty RLS structure and let M be a subset of V. We say that M is convex if and only if:

(Def. 2) For all vectors u, v of V and for every real number r such that $0 < r$ and $r < 1$ and $u \in M$ and $v \in M$ holds $r \cdot u + (1 - r) \cdot v \in M$.

Next we state a number of propositions:

(1) Let V be a real linear space-like non empty RLS structure, M be a subset of V, and r be a real number. If M is convex, then $r \cdot M$ is convex.

(2) Let V be an Abelian add-associative real linear space-like non empty RLS structure and M, N be subsets of V. If M is convex and N is convex, then $M + N$ is convex.

(3) For every real linear space V and for all subsets M, N of V such that M is convex and N is convex holds $M - N$ is convex.

(4) Let V be a non empty RLS structure and M be a subset of V. Then M is convex if and only if for every real number r such that $0 < r$ and $r < 1$ holds $r \cdot M + (1-r) \cdot M \subseteq M$.

(5) Let V be an Abelian non empty RLS structure and M be a subset of V. Suppose M is convex. Let r be a real number. If $0 < r$ and $r < 1$, then $(1-r) \cdot M + r \cdot M \subseteq M$.

(6) Let V be an Abelian add-associative real linear space-like non empty RLS structure and M, N be subsets of V. Suppose M is convex and N is convex. Let r be a real number. Then $r \cdot M + (1-r) \cdot N$ is convex.

(7) Let V be a real linear space, M be a subset of V, and v be a vector of V. Then M is convex if and only if $v + M$ is convex.

(8) For every real linear space V holds $\mathrm{Up}(\mathbf{0}_V)$ is convex.

(9) For every real linear space V holds $\mathrm{Up}(\Omega_V)$ is convex.

(10) For every non empty RLS structure V and for every subset M of V such that $M = \emptyset$ holds M is convex.

(11) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M_1, M_2 be subsets of V, and r_1, r_2 be real numbers. If M_1 is convex and M_2 is convex, then $r_1 \cdot M_1 + r_2 \cdot M_2$ is convex.

(12) Let V be a real linear space-like non empty RLS structure, M be a subset of V, and r_1, r_2 be real numbers. Then $(r_1 + r_2) \cdot M \subseteq r_1 \cdot M + r_2 \cdot M$.

(13) Let V be a real linear space, M be a subset of V, and r_1, r_2 be real numbers. If $r_1 \geq 0$ and $r_2 \geq 0$ and M is convex, then $r_1 \cdot M + r_2 \cdot M \subseteq (r_1 + r_2) \cdot M$.

(14) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M_1, M_2, M_3 be subsets of V, and r_1, r_2, r_3 be real numbers. If M_1 is convex and M_2 is convex and M_3 is convex, then $r_1 \cdot M_1 + r_2 \cdot M_2 + r_3 \cdot M_3$ is convex.

(15) Let V be a non empty RLS structure and F be a family of subsets of V. Suppose that for every subset M of V such that $M \in F$ holds M is convex. Then $\bigcap F$ is convex.

(16) For every non empty RLS structure V and for every subset M of V such that M is Affine holds M is convex.

Let V be a non empty RLS structure. One can check that there exists a subset of V which is convex.

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Let V be a non empty RLS structure. One can check that there exists a subset of V which is non empty and convex.

We now state four propositions:

(17) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) \geq r\}$, then M is convex.

(18) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) > r\}$, then M is convex.

(19) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) \leq r\}$, then M is convex.

(20) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) < r\}$, then M is convex.

2. Convex Combinations

Let V be a real linear space and let L be a linear combination of V. We say that L is convex if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a finite sequence F of elements of the carrier of V such that

(i) F is one-to-one,

(ii) $\operatorname{rng} F =$ the support of L, and

(iii) there exists a finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f = \operatorname{len} F$ and $\sum f = 1$ and for every natural number n such that $n \in \operatorname{dom} f$ holds $f(n) = L(F(n))$ and $f(n) \geq 0$.

The following propositions are true:

(21) Let V be a real linear space and L be a linear combination of V. If L is convex, then the support of $L \neq \emptyset$.

(22) Let V be a real linear space, L be a linear combination of V, and v be a vector of V. If L is convex and $L(v) \leq 0$, then $v \notin$ the support of L.

(23) For every real linear space V and for every linear combination L of V such that L is convex holds $L \neq \mathbf{0}_{\mathrm{LC}_V}$.

(24) Let V be a real linear space, v be a vector of V, and L be a linear combination of $\{v\}$. If L is convex, then $L(v) = 1$ and $\sum L = L(v) \cdot v$.

(25) Let V be a real linear space, v_1, v_2 be vectors of V, and L be a linear combination of $\{v_1, v_2\}$. Suppose $v_1 \neq v_2$ and L is convex. Then $L(v_1) + L(v_2) = 1$ and $L(v_1) \geq 0$ and $L(v_2) \geq 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

(26) Let V be a real linear space, v_1, v_2, v_3 be vectors of V, and L be a linear combination of $\{v_1, v_2, v_3\}$. Suppose $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_3 \neq v_1$ and L is convex. Then $L(v_1) + L(v_2) + L(v_3) = 1$ and $L(v_1) \geq 0$ and $L(v_2) \geq 0$ and $L(v_3) \geq 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.

(27) Let V be a real linear space, v be a vector of V, and L be a linear combination of V. If L is convex and the support of $L = \{v\}$, then $L(v) = 1$.

(28) Let V be a real linear space, v_1, v_2 be vectors of V, and L be a linear combination of V. Suppose L is convex and the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$. Then $L(v_1) + L(v_2) = 1$ and $L(v_1) \geq 0$ and $L(v_2) \geq 0$.

(29) Let V be a real linear space, v_1, v_2, v_3 be vectors of V, and L be a linear combination of V. Suppose L is convex and the support of $L = \{v_1, v_2, v_3\}$ and $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_3 \neq v_1$. Then $L(v_1) + L(v_2) + L(v_3) = 1$ and $L(v_1) \geq 0$ and $L(v_2) \geq 0$ and $L(v_3) \geq 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.

3. Convex Hull

In this article we present several logical schemes. The scheme *SubFamExRLS* deals with an RLS structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

The scheme *SubFamExRLS2* deals with an RLS structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

Let V be a non empty RLS structure and let M be a subset of V. The functor Convex-Family M yields a family of subsets of V and is defined by:

(Def. 4) For every subset N of V holds $N \in \text{Convex-Family}\, M$ iff N is convex and $M \subseteq N$.

Let V be a non empty RLS structure and let M be a subset of V. The functor $\text{conv}\, M$ yields a convex subset of V and is defined by:

(Def. 5) $\text{conv}\, M = \bigcap \text{Convex-Family}\, M$.

The following proposition is true

(30) Let V be a non empty RLS structure, M be a subset of V, and N be a convex subset of V. If $M \subseteq N$, then $\text{conv}\, M \subseteq N$.

4. Miscellaneous

The following propositions are true:

(31) Let p be a finite sequence and x, y, z be sets. Suppose p is one-to-one and $\text{rng}\, p = \{x, y, z\}$ and $x \neq y$ and $y \neq z$ and $z \neq x$. Then $p = \langle x, y, z \rangle$ or $p = \langle x, z, y \rangle$ or $p = \langle y, x, z \rangle$ or $p = \langle y, z, x \rangle$ or $p = \langle z, x, y \rangle$ or $p = \langle z, y, x \rangle$.

(32) For every real linear space-like non empty RLS structure V and for every subset M of V holds $1 \cdot M = M$.

(33) For every non empty RLS structure V and for every empty subset M of V and for every real number r holds $r \cdot M = \emptyset$.

(34) For every real linear space V and for every non empty subset M of V holds $0 \cdot M = \{0_V\}$.

(35) For every right zeroed non empty loop structure V and for every subset M of V holds $M + \{0_V\} = M$.

(36) For every add-associative non empty loop structure V and for all subsets M_1, M_2, M_3 of V holds $(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$.

(37) Let V be a real linear space-like non empty RLS structure, M be a subset of V, and r_1, r_2 be real numbers. Then $r_1 \cdot (r_2 \cdot M) = (r_1 \cdot r_2) \cdot M$.

(38) Let V be a real linear space-like non empty RLS structure, M_1, M_2 be subsets of V, and r be a real number. Then $r \cdot (M_1 + M_2) = r \cdot M_1 + r \cdot M_2$.

(39) Let V be a non empty RLS structure, M, N be subsets of V, and r be a real number. If $M \subseteq N$, then $r \cdot M \subseteq r \cdot N$.

(40) For every non empty loop structure V and for every empty subset M of V and for every subset N of V holds $M + N = \emptyset$.

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