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The Set of Primitive Recursive Functions[1]

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Summary. We follow [31] in defining the set of primitive recursive functions. The important helper notion is the homogeneous function from finite sequences of natural numbers into natural numbers where homogeneous means that all the sequences in the domain are of the same length. The set of all such functions is then used to define the notion of a set closed under composition of functions and under primitive recursion. We call a set primitively recursively closed iff it contains the initial functions (nullary constant function returning 0, unary successor and projection functions for all arities) and is closed under composition and primitive recursion. The set of primitive recursive functions is then defined as the smallest set of functions which is primitive recursively closed. We show that this set can be obtained by primitive recursive approximation. We finish with showing that some simple and well known functions are primitive recursive.

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The articles [25], [11], [30], [27], [1], [32], [33], [8], [6], [12], [24], [16], [26], [9], [13], [3], [22], [4], [21], [29], [10], [19], [15], [18], [5], [7], [14], [28], [20], [17], [23], and [2] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: i, j, k, c, m, n denote natural numbers, a, x, y, z, X, Y denote sets, D, E denote non empty sets, R denotes a binary relation, f, g denote functions, and p, q denote finite sequences.

Let X be a non empty set, let n be a natural number, let p be an element of X^n, let i be a natural number, and let x be an element of X. Then $p +\cdot (i,x)$ is an element of X^n.

Let n be a natural number, let t be an element of $\mathbb{N}^n$, and let i be a natural number. Then $t(i)$ is an element of $\mathbb{N}$.

The following propositions are true:

(3)[1] $\langle x,y\rangle +\cdot (1,z) = \langle z,y\rangle$ and $\langle x,y\rangle +\cdot (2,z) = \langle x,z\rangle$.

(5)[2] If $f +\cdot (a,x) = g +\cdot (a,y)$, then $f +\cdot (a,z) = g +\cdot (a,z)$.

(6) $(p +\cdot (i,x))_{\restriction i} = p_{\restriction i}$.

(7) If $p +\cdot (i,a) = q +\cdot (i,a)$, then $p_{\restriction i} = q_{\restriction i}$.

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[1] The propositions (1) and (2) have been removed.

[2] The proposition (4) has been removed.

(8) $X^0 = \{\emptyset\}$.

(9) If $n \neq 0$, then $\emptyset^n = \emptyset$.

(10) If $\emptyset \in \operatorname{rng} f$, then $\prod^* f = \emptyset$.

(11) If $\operatorname{rng} f = D$, then $\operatorname{rng} \prod^* \langle f \rangle = D^1$.

(12) If $1 \leq i$ and $i \leq n+1$, then for every element p of D^{n+1} holds $p_{\restriction i} \in D^n$.

(13) For every set X and for every set Y of finite sequences of X holds $Y \subseteq X^*$.

2. Sets of Compatible Functions

Let X be a set. We say that X is compatible if and only if:

(Def. 1) For all functions f, g such that $f \in X$ and $g \in X$ holds $f \approx g$.

Let us observe that there exists a set which is non empty, functional, and compatible.
Let X be a functional compatible set. One can check that $\bigcup X$ is function-like and relation-like.
Next we state the proposition

(14) X is functional and compatible iff $\bigcup X$ is a function.

Let X, Y be sets. Note that there exists a non empty set of partial functions from X to Y which is non empty and compatible.
One can prove the following propositions:

(15) For every non empty functional compatible set X holds $\operatorname{dom} \bigcup X = \bigcup \{\operatorname{dom} f : f$ ranges over elements of $X\}$.

(16) Let X be a functional compatible set and f be a function. If $f \in X$, then $\operatorname{dom} f \subseteq \operatorname{dom} \bigcup X$ and for every set x such that $x \in \operatorname{dom} f$ holds $(\bigcup X)(x) = f(x)$.

(17) For every non empty functional compatible set X holds $\operatorname{rng} \bigcup X = \bigcup \{\operatorname{rng} f : f$ ranges over elements of $X\}$.

Let us consider X, Y. Note that every non empty set of partial functions from X to Y is functional.
The following proposition is true

(18) Let P be a compatible non empty set of partial functions from X to Y. Then $\bigcup P$ is a partial function from X to Y.

3. Homogeneous Relations

Let f be a binary relation. We introduce f is into $\mathbb{N}$ as a synonym of f is natural-yielding.
Let f be a binary relation. We say that f is from tuples on $\mathbb{N}$ if and only if:

(Def. 2) $\operatorname{dom} f \subseteq \mathbb{N}^*$.

Let us note that there exists a function which is from tuples on $\mathbb{N}$ and into $\mathbb{N}$.
Let f be a binary relation from tuples on $\mathbb{N}$. We say that f is length total if and only if:

(Def. 3) For all finite sequences x, y of elements of $\mathbb{N}$ such that $\operatorname{len} x = \operatorname{len} y$ and $x \in \operatorname{dom} f$ holds $y \in \operatorname{dom} f$.

Let f be a binary relation. We say that f is homogeneous if and only if:

(Def. 4) For all finite sequences x, y such that $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} f$ holds $\operatorname{len} x = \operatorname{len} y$.

One can prove the following proposition

(19) If $\operatorname{dom} R \subseteq D^n$, then R is homogeneous.

One can check that $\emptyset$ is homogeneous.

Let p be a finite sequence and let x be a set. Observe that $\{p\} \longmapsto x$ is non empty and homogeneous.

One can verify that there exists a function which is non empty and homogeneous.

Let f be a homogeneous function and let g be a function. One can verify that $g \cdot f$ is homogeneous.

Let X, Y be sets. One can check that there exists a partial function from X^* to Y which is homogeneous.

Let X, Y be non empty sets. One can check that there exists a partial function from X^* to Y which is non empty and homogeneous.

Let X be a non empty set. Observe that there exists a partial function from X^* to X which is non empty, homogeneous, and quasi total.

Let us observe that there exists a function from tuples on $\mathbb{N}$ which is non empty, homogeneous, into $\mathbb{N}$, and length total.

Let us mention that every partial function from $\mathbb{N}^*$ to $\mathbb{N}$ is into $\mathbb{N}$ and from tuples on $\mathbb{N}$.

Let us observe that every partial function from $\mathbb{N}^*$ to $\mathbb{N}$ which is quasi total is also length total.

Next we state the proposition

(20) Every length total function from tuples on $\mathbb{N}$ into $\mathbb{N}$ is a quasi total partial function from $\mathbb{N}^*$ to $\mathbb{N}$.

Let f be a homogeneous binary relation. The functor $\operatorname{arity} f$ yields a natural number and is defined by:

(Def. 5)(i) For every finite sequence x such that $x \in \operatorname{dom} f$ holds $\operatorname{arity} f = \operatorname{len} x$ if there exists a finite sequence x such that $x \in \operatorname{dom} f$,

(ii) $\operatorname{arity} f = 0$, otherwise.

Next we state several propositions:

(21) $\operatorname{arity} \emptyset = 0$.

(22) For every homogeneous binary relation f such that $\operatorname{dom} f = \{\emptyset\}$ holds $\operatorname{arity} f = 0$.

(23) For every homogeneous partial function f from X^* to Y holds $\operatorname{dom} f \subseteq X^{\operatorname{arity} f}$.

(24) For every homogeneous function f from tuples on $\mathbb{N}$ holds $\operatorname{dom} f \subseteq \mathbb{N}^{\operatorname{arity} f}$.

(25) Let f be a homogeneous partial function from X^* to X. Then f is quasi total and non empty if and only if $\operatorname{dom} f = X^{\operatorname{arity} f}$.

(26) Let f be a homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$. Then f is length total and non empty if and only if $\operatorname{dom} f = \mathbb{N}^{\operatorname{arity} f}$.

(27) For every non empty homogeneous partial function f from D^* to D and for every n such that $\operatorname{dom} f \subseteq D^n$ holds $\operatorname{arity} f = n$.

(28) For every homogeneous partial function f from D^* to D and for every n such that $\operatorname{dom} f = D^n$ holds $\operatorname{arity} f = n$.

Let R be a binary relation. We say that R has the same arity if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let f, g be functions such that $f \in \operatorname{rng} R$ and $g \in \operatorname{rng} R$. Then

(i) if f is empty, then g is empty or $\operatorname{dom} g = \{\emptyset\}$, and

(ii) if f is non empty and g is non empty, then there exists a natural number n and there exists a non empty set X such that $\operatorname{dom} f \subseteq X^n$ and $\operatorname{dom} g \subseteq X^n$.

Let us mention that $\emptyset$ has the same arity.

Observe that there exists a finite sequence which has the same arity. Let X be a set. One can check that there exists a finite sequence of elements of X which has the same arity and there exists an element of X^* which has the same arity.

Let F be a binary relation with the same arity. The functor arity F yields a natural number and is defined by:

(Def. 7)(i) For every homogeneous function f such that $f \in \operatorname{rng} F$ holds $\operatorname{arity} F = \operatorname{arity} f$ if there exists a homogeneous function f such that $f \in \operatorname{rng} F$,

(ii) $\operatorname{arity} F = 0$, otherwise.

The following proposition is true

(29) For every finite sequence F with the same arity such that $\operatorname{len} F = 0$ holds $\operatorname{arity} F = 0$.

Let X be a set. The functor HFuncs X yields a non empty set of partial functions from X^* to X and is defined as follows:

(Def. 8) $\mathrm{HFuncs}\, X = \{f; f$ ranges over elements of $X^* \dot{\to} X : f$ is homogeneous$\}$.

We now state the proposition

(30) $\emptyset \in \mathrm{HFuncs}\, X$.

Let X be a non empty set. Observe that there exists an element of HFuncs X which is non empty, homogeneous, and quasi total.

Let X be a set. Observe that every element of HFuncs X is homogeneous.

Let X be a non empty set and let S be a non empty subset of HFuncs X. Note that every element of S is homogeneous.

The following propositions are true:

(31) Every homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$ is an element of HFuncs $\mathbb{N}$.

(32) Every length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$ is a quasi total element of HFuncs $\mathbb{N}$.

(33) Let X be a non empty set and F be a binary relation such that $\operatorname{rng} F \subseteq \mathrm{HFuncs}\, X$ and for all homogeneous functions f, g such that $f \in \operatorname{rng} F$ and $g \in \operatorname{rng} F$ holds $\operatorname{arity} f = \operatorname{arity} g$. Then F has the same arity.

Let n, m be natural numbers. The functor $\mathrm{const}_n(m)$ yielding a homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$ is defined by:

(Def. 9) $\mathrm{const}_n(m) = \mathbb{N}^n \longmapsto m$.

One can prove the following proposition

(34) $\mathrm{const}_n(m) \in \mathrm{HFuncs}\, \mathbb{N}$.

Let n, m be natural numbers. Note that $\mathrm{const}_n(m)$ is length total and non empty.

One can prove the following propositions:

(35) $\operatorname{arity} \mathrm{const}_n(m) = n$.

(36) For every element t of $\mathbb{N}^n$ holds $(\mathrm{const}_n(m))(t) = m$.

Let n, i be natural numbers. The functor $\mathrm{succ}_n(i)$ yields a homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$ and is defined as follows:

(Def. 10) $\operatorname{dom} \mathrm{succ}_n(i) = \mathbb{N}^n$ and for every element p of $\mathbb{N}^n$ holds $(\mathrm{succ}_n(i))(p) = p_i + 1$.

The following proposition is true

(37) $\mathrm{succ}_n(i) \in \mathrm{HFuncs}\,\mathbb{N}$.

Let n, i be natural numbers. One can check that $\mathrm{succ}_n(i)$ is length total and non empty.
The following proposition is true

(38) $\mathrm{arity}\,\mathrm{succ}_n(i) = n$.

Let n, i be natural numbers. The functor $\mathrm{proj}_n(i)$ yields a homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$ and is defined as follows:

(Def. 11) $\mathrm{proj}_n(i) = \mathrm{proj}(n \mapsto \mathbb{N}, i)$.

We now state two propositions:

(39) $\mathrm{proj}_n(i) \in \mathrm{HFuncs}\,\mathbb{N}$.

(40) $\mathrm{dom}\,\mathrm{proj}_n(i) = \mathbb{N}^n$ and if $1 \leq i$ and $i \leq n$, then $\mathrm{rng}\,\mathrm{proj}_n(i) = \mathbb{N}$.

Let n, i be natural numbers. Observe that $\mathrm{proj}_n(i)$ is length total and non empty.
One can prove the following two propositions:

(41) $\mathrm{arity}\,\mathrm{proj}_n(i) = n$.

(42) For every element t of $\mathbb{N}^n$ holds $(\mathrm{proj}_n(i))(t) = t(i)$.

Let X be a set. One can check that $\mathrm{HFuncs}\,X$ is functional.
Next we state three propositions:

(43) Let F be a function from D into $\mathrm{HFuncs}\,E$. Suppose $\mathrm{rng}\,F$ is compatible and for every element x of D holds $\mathrm{dom}\,F(x) \subseteq E^n$. Then there exists an element f of $\mathrm{HFuncs}\,E$ such that $f = \bigcup F$ and $\mathrm{dom}\,f \subseteq E^n$.

(44) For every function F from $\mathbb{N}$ into $\mathrm{HFuncs}\,D$ such that for every i holds $F(i) \subseteq F(i+1)$ holds $\bigcup F \in \mathrm{HFuncs}\,D$.

(45) For every finite sequence F of elements of $\mathrm{HFuncs}\,D$ with the same arity holds $\mathrm{dom}\prod^* F \subseteq D^{\mathrm{arity}\,F}$.

Let X be a non empty set and let F be a finite sequence of elements of $\mathrm{HFuncs}\,X$ with the same arity. Observe that $\prod^* F$ is homogeneous.
Next we state the proposition

(46) Let f be an element of $\mathrm{HFuncs}\,D$ and F be a finite sequence of elements of $\mathrm{HFuncs}\,D$ with the same arity. Then $\mathrm{dom}(f \cdot \prod^* F) \subseteq D^{\mathrm{arity}\,F}$ and $\mathrm{rng}(f \cdot \prod^* F) \subseteq D$ and $f \cdot \prod^* F \in \mathrm{HFuncs}\,D$.

Let X, Y be non empty sets, let P be a non empty set of partial functions from X to Y, and let S be a non empty subset of P. We see that the element of S is an element of P.
Let f be a homogeneous function from tuples on $\mathbb{N}$. Observe that $\langle f \rangle$ has the same arity.
We now state several propositions:

(47) For every homogeneous function f into $\mathbb{N}$ and from tuples on $\mathbb{N}$ holds $\mathrm{arity}\langle f \rangle = \mathrm{arity}\,f$.

(48) Let f, g be non empty elements of $\mathrm{HFuncs}\,\mathbb{N}$ and F be a finite sequence of elements of $\mathrm{HFuncs}\,\mathbb{N}$ with the same arity. If $g = f \cdot \prod^* F$, then $\mathrm{arity}\,g = \mathrm{arity}\,F$.

(49) Let f be a non empty quasi total element of $\mathrm{HFuncs}\,D$ and F be a finite sequence of elements of $\mathrm{HFuncs}\,D$ with the same arity. Suppose $\mathrm{arity}\,f = \mathrm{len}\,F$ and F is non empty and for every element h of $\mathrm{HFuncs}\,D$ such that $h \in \mathrm{rng}\,F$ holds h is quasi total and non empty. Then $f \cdot \prod^* F$ is a non empty quasi total element of $\mathrm{HFuncs}\,D$ and $\mathrm{dom}(f \cdot \prod^* F) = D^{\mathrm{arity}\,F}$.

(50) Let f be a quasi total element of $\mathrm{HFuncs}\, D$ and F be a finite sequence of elements of $\mathrm{HFuncs}\, D$ with the same arity. Suppose $\mathrm{arity}\, f = \mathrm{len}\, F$ and for every element h of $\mathrm{HFuncs}\, D$ such that $h \in \mathrm{rng}\, F$ holds h is quasi total. Then $f \cdot \prod^* F$ is a quasi total element of $\mathrm{HFuncs}\, D$.

(51) For all non empty quasi total elements f, g of $\mathrm{HFuncs}\, D$ such that $\mathrm{arity}\, f = 0$ and $\mathrm{arity}\, g = 0$ and $f(\emptyset) = g(\emptyset)$ holds $f = g$.

(52) Let f, g be non empty length total homogeneous functions from tuples on $\mathbb{N}$ into $\mathbb{N}$. If $\mathrm{arity}\, f = 0$ and $\mathrm{arity}\, g = 0$ and $f(\emptyset) = g(\emptyset)$, then $f = g$.

4. Primitive Recursiveness

We adopt the following convention: f_1, f_2 denote non empty homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$, e_1, e_2 denote homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$, and p denotes an element of $\mathbb{N}^{\mathrm{arity}\, f_1 + 1}$.

Let g, f_1, f_2 be homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$ and let i be a natural number. We say that g is primitive recursively expressed by f_1, f_2 and i if and only if the condition (Def. 12) is satisfied.

(Def. 12) There exists a natural number n such that

(i) $\mathrm{dom}\, g \subseteq \mathbb{N}^n$,

(ii) $i \geq 1$,

(iii) $i \leq n$,

(iv) $\mathrm{arity}\, f_1 + 1 = n$,

(v) $n + 1 = \mathrm{arity}\, f_2$, and

(vi) for every finite sequence p of elements of $\mathbb{N}$ such that $\mathrm{len}\, p = n$ holds $p +\cdot (i, 0) \in \mathrm{dom}\, g$ iff $p_{\restriction i} \in \mathrm{dom}\, f_1$ and if $p +\cdot (i, 0) \in \mathrm{dom}\, g$, then $g(p +\cdot (i, 0)) = f_1(p_{\restriction i})$ and for every natural number n holds $p +\cdot (i, n+1) \in \mathrm{dom}\, g$ iff $p +\cdot (i, n) \in \mathrm{dom}\, g$ and $(p +\cdot (i, n)) ^\frown \langle g(p +\cdot (i, n)) \rangle \in \mathrm{dom}\, f_2$ and if $p +\cdot (i, n+1) \in \mathrm{dom}\, g$, then $g(p +\cdot (i, n+1)) = f_2((p +\cdot (i, n)) ^\frown \langle g(p +\cdot (i, n)) \rangle)$.

Let f_1, f_2 be homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$, let i be a natural number, and let p be a finite sequence of elements of $\mathbb{N}$. The functor $\mathrm{primrec}(f_1, f_2, i, p)$ yields an element of $\mathrm{HFuncs}\, \mathbb{N}$ and is defined by the condition (Def. 13).

(Def. 13) There exists a function F from $\mathbb{N}$ into $\mathrm{HFuncs}\, \mathbb{N}$ such that

(i) $\mathrm{primrec}(f_1, f_2, i, p) = F(p_i)$,

(ii) if $i \in \mathrm{dom}\, p$ and $p_{\restriction i} \in \mathrm{dom}\, f_1$, then $F(0) = \{p +\cdot (i, 0)\} \longmapsto f_1(p_{\restriction i})$,

(iii) if $i \notin \mathrm{dom}\, p$ or $p_{\restriction i} \notin \mathrm{dom}\, f_1$, then $F(0) = \emptyset$, and

(iv) for every natural number m holds if $i \in \mathrm{dom}\, p$ and $p +\cdot (i, m) \in \mathrm{dom}\, F(m)$ and $(p +\cdot (i, m)) ^\frown \langle F(m)(p +\cdot (i, m)) \rangle \in \mathrm{dom}\, f_2$, then $F(m+1) = F(m) +\cdot (\{p +\cdot (i, m+1)\} \longmapsto f_2((p +\cdot (i, m)) ^\frown \langle F(m)(p +\cdot (i, m)) \rangle))$ and if $i \notin \mathrm{dom}\, p$ or $p +\cdot (i, m) \notin \mathrm{dom}\, F(m)$ or $(p +\cdot (i, m)) ^\frown \langle F(m)(p +\cdot (i, m)) \rangle \notin \mathrm{dom}\, f_2$, then $F(m+1) = F(m)$.

The following propositions are true:

(53) For all finite sequences p, q of elements of $\mathbb{N}$ such that $q \in \mathrm{dom}\, \mathrm{primrec}(e_1, e_2, i, p)$ there exists k such that $q = p +\cdot (i, k)$.

(54) For every finite sequence p of elements of $\mathbb{N}$ such that $i \notin \mathrm{dom}\, p$ holds $\mathrm{primrec}(e_1, e_2, i, p) = \emptyset$.

(55) For all finite sequences p, q of elements of $\mathbb{N}$ holds $\mathrm{primrec}(e_1, e_2, i, p) \approx \mathrm{primrec}(e_1, e_2, i, q)$.

(56) For every finite sequence p of elements of $\mathbb{N}$ holds $\mathrm{dom}\, \mathrm{primrec}(e_1, e_2, i, p) \subseteq \mathbb{N}^{1 + \mathrm{arity}\, e_1}$.

(57) For every finite sequence p of elements of $\mathbb{N}$ such that e_1 is empty holds $\mathrm{primrec}(e_1, e_2, i, p)$ is empty.

(58) If f_1 is length total and f_2 is length total and $\mathrm{arity}\, f_1 + 2 = \mathrm{arity}\, f_2$ and $1 \leq i$ and $i \leq 1 + \mathrm{arity}\, f_1$, then $p \in \mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i, p)$.

Let f_1, f_2 be homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$ and let i be a natural number. The functor $\mathrm{primrec}(f_1, f_2, i)$ yielding an element of HFuncs $\mathbb{N}$ is defined as follows:

(Def. 14) There exists a function G from $\mathbb{N}^{\mathrm{arity}\, f_1 + 1}$ into HFuncs $\mathbb{N}$ such that $\mathrm{primrec}(f_1, f_2, i) = \bigcup G$ and for every element p of $\mathbb{N}^{\mathrm{arity}\, f_1 + 1}$ holds $G(p) = \mathrm{primrec}(f_1, f_2, i, p)$.

We now state a number of propositions:

(59) If e_1 is empty, then $\mathrm{primrec}(e_1, e_2, i)$ is empty.

(60) $\mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i) \subseteq \mathbb{N}^{\mathrm{arity}\, f_1 + 1}$.

(61) If f_1 is length total and f_2 is length total and $\mathrm{arity}\, f_1 + 2 = \mathrm{arity}\, f_2$ and $1 \leq i$ and $i \leq 1 + \mathrm{arity}\, f_1$, then $\mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i) = \mathbb{N}^{\mathrm{arity}\, f_1 + 1}$ and $\mathrm{arity}\,\mathrm{primrec}(f_1, f_2, i) = \mathrm{arity}\, f_1 + 1$.

(62) If $i \in \mathrm{dom}\, p$, then $p + \cdot\, (i, 0) \in \mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i)$ iff $p_{\restriction i} \in \mathrm{dom}\, f_1$.

(63) If $i \in \mathrm{dom}\, p$ and $p + \cdot\, (i, 0) \in \mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i)$, then $(\mathrm{primrec}(f_1, f_2, i))(p + \cdot\, (i, 0)) = f_1(p_{\restriction i})$.

(64) If $i \in \mathrm{dom}\, p$ and f_1 is length total, then $(\mathrm{primrec}(f_1, f_2, i))(p + \cdot\, (i, 0)) = f_1(p_{\restriction i})$.

(65) If $i \in \mathrm{dom}\, p$, then $p + \cdot\, (i, m+1) \in \mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i)$ iff $p + \cdot\, (i, m) \in \mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i)$ and $(p + \cdot\, (i, m)) \frown \langle (\mathrm{primrec}(f_1, f_2, i))(p + \cdot\, (i, m)) \rangle \in \mathrm{dom}\, f_2$.

(66) If $i \in \mathrm{dom}\, p$ and $p + \cdot\, (i, m+1) \in \mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i)$, then $(\mathrm{primrec}(f_1, f_2, i))(p + \cdot\, (i, m+1)) = f_2((p + \cdot\, (i, m)) \frown \langle (\mathrm{primrec}(f_1, f_2, i))(p + \cdot\, (i, m)) \rangle)$.

(67) Suppose f_1 is length total and f_2 is length total and $\mathrm{arity}\, f_1 + 2 = \mathrm{arity}\, f_2$ and $1 \leq i$ and $i \leq 1 + \mathrm{arity}\, f_1$. Then $(\mathrm{primrec}(f_1, f_2, i))(p + \cdot\, (i, m+1)) = f_2((p + \cdot\, (i, m)) \frown \langle (\mathrm{primrec}(f_1, f_2, i))(p + \cdot\, (i, m)) \rangle)$.

(68) If $\mathrm{arity}\, f_1 + 2 = \mathrm{arity}\, f_2$ and $1 \leq i$ and $i \leq \mathrm{arity}\, f_1 + 1$, then $\mathrm{primrec}(f_1, f_2, i)$ is primitive recursively expressed by f_1, f_2 and i.

(69) Suppose $1 \leq i$ and $i \leq \mathrm{arity}\, f_1 + 1$. Let g be an element of HFuncs $\mathbb{N}$. If g is primitive recursively expressed by f_1, f_2 and i, then $g = \mathrm{primrec}(f_1, f_2, i)$.

5. The Set of Primitive Recursive Functions

Let X be a set. We say that X is composition closed if and only if the condition (Def. 15) is satisfied.

(Def. 15) Let f be an element of HFuncs $\mathbb{N}$ and F be a finite sequence of elements of HFuncs $\mathbb{N}$ with the same arity. If $f \in X$ and $\mathrm{arity}\, f = \mathrm{len}\, F$ and $\mathrm{rng}\, F \subseteq X$, then $f \cdot \prod^* F \in X$.

We say that X is primitive recursion closed if and only if the condition (Def. 16) is satisfied.

(Def. 16) Let g, f_1, f_2 be elements of HFuncs $\mathbb{N}$ and i be a natural number. Suppose g is primitive recursively expressed by f_1, f_2 and i and $f_1 \in X$ and $f_2 \in X$. Then $g \in X$.

Let X be a set. We say that X is primitive recursively closed if and only if the conditions (Def. 17) are satisfied.

(Def. 17)(i) $\mathrm{const}_0(0) \in X$,

(ii) $\mathrm{succ}_1(1) \in X$,

(iii) for all natural numbers n, i such that $1 \leq i$ and $i \leq n$ holds $\mathrm{proj}_n(i) \in X$, and

(iv) X is composition closed and primitive recursion closed.

Next we state the proposition

(70) HFuncs $\mathbb{N}$ is primitive recursively closed.

Let us note that there exists a subset of HFuncs $\mathbb{N}$ which is primitive recursively closed and non empty.

In the sequel P is a primitive recursively closed non empty subset of HFuncs $\mathbb{N}$.

The following propositions are true:

(71) For every element g of HFuncs $\mathbb{N}$ such that $e_1 = \emptyset$ and g is primitive recursively expressed by e_1, e_2 and i holds $g = \emptyset$.

(72) Let g be an element of HFuncs $\mathbb{N}$, f_1, f_2 be quasi total elements of HFuncs $\mathbb{N}$, and i be a natural number. Suppose g is primitive recursively expressed by f_1, f_2 and i. Then g is quasi total and if f_1 is non empty, then g is non empty.

(73) $\mathrm{const}_n(c) \in P$.

(74) If $1 \leq i$ and $i \leq n$, then $\mathrm{succ}_n(i) \in P$.

(75) $\emptyset \in P$.

(76) Let f be an element of P and F be a finite sequence of elements of P with the same arity. If arity $f = \mathrm{len}\, F$, then $f \cdot \prod^* F \in P$.

(77) Let f_1, f_2 be elements of P. Suppose arity $f_1 + 2 =$ arity f_2. Let i be a natural number. If $1 \leq i$ and $i \leq$ arity $f_1 + 1$, then $\mathrm{primrec}(f_1, f_2, i) \in P$.

The subset PrimRec of HFuncs $\mathbb{N}$ is defined as follows:

(Def. 18) $\mathrm{PrimRec} = \bigcap\{R; R$ ranges over elements of $2^{\mathrm{HFuncs}\,\mathbb{N}}$: R is primitive recursively closed$\}$.

One can prove the following proposition

(78) For every subset X of HFuncs $\mathbb{N}$ such that X is primitive recursively closed holds $\mathrm{PrimRec} \subseteq X$.

One can verify that PrimRec is non empty and primitive recursively closed.

Let us observe that every element of PrimRec is homogeneous.

Let x be a set. We say that x is primitive recursive if and only if:

(Def. 19) $x \in \mathrm{PrimRec}$.

Let us observe that every set which is primitive recursive is also relation-like and function-like.

Let us observe that every binary relation which is primitive recursive is also homogeneous, into $\mathbb{N}$, and from tuples on $\mathbb{N}$.

Let us note that every element of PrimRec is primitive recursive.

One can check that there exists a function which is primitive recursive and there exists an element of HFuncs $\mathbb{N}$ which is primitive recursive.

The initial functions constitute a subset of HFuncs $\mathbb{N}$ defined by:

(Def. 20) The initial functions $= \{\mathrm{const}_0(0), \mathrm{succ}_1(1)\} \cup \{\mathrm{proj}_n(i); n$ ranges over natural numbers, i ranges over natural numbers: $1 \leq i \wedge i \leq n\}$.

Let Q be a subset of HFuncs $\mathbb{N}$. The primitive recursion closure of Q is a subset of HFuncs $\mathbb{N}$ and is defined by the condition (Def. 21).

(Def. 21) The primitive recursion closure of $Q = Q \cup \{g; g$ ranges over elements of HFuncs $\mathbb{N}$: $\bigvee_{f_1, f_2 : \text{element of HFuncs}\,\mathbb{N}} \bigvee_{i : \text{natural number}} (f_1 \in Q \wedge f_2 \in Q \wedge g$ is primitive recursively expressed by f_1, f_2 and $i)\}$.

The composition closure of Q is a subset of HFuncs $\mathbb{N}$ and is defined by the condition (Def. 22).

(Def. 22) The composition closure of $Q = Q \cup \{f \cdot \prod^* F; f$ ranges over elements of HFuncs$\,\mathbb{N}$, F ranges over elements of $(\mathrm{HFuncs}\,\mathbb{N})^*$ with the same arity: $f \in Q \wedge \mathrm{arity}\, f = \mathrm{len}\, F \wedge \mathrm{rng}\, F \subseteq Q\}$.

The function $\mathrm{PrimRec}^{\approx}$ from $\mathbb{N}$ into $2^{\mathrm{HFuncs}\,\mathbb{N}}$ is defined by the conditions (Def. 23).

(Def. 23)(i) $\mathrm{PrimRec}^{\approx}(0) =$ the initial functions, and

(ii) for every natural number m holds $\mathrm{PrimRec}^{\approx}(m+1) =$ (the primitive recursion closure of $\mathrm{PrimRec}^{\approx}(m)) \cup$ (the composition closure of $\mathrm{PrimRec}^{\approx}(m)$).

Next we state four propositions:

(79) If $m \leq n$, then $\mathrm{PrimRec}^{\approx}(m) \subseteq \mathrm{PrimRec}^{\approx}(n)$.

(80) $\bigcup(\mathrm{PrimRec}^{\approx})$ is primitive recursively closed.

(81) $\mathrm{PrimRec} = \bigcup(\mathrm{PrimRec}^{\approx})$.

(82) For every element f of HFuncs$\,\mathbb{N}$ such that $f \in \mathrm{PrimRec}^{\approx}(m)$ holds f is quasi total.

One can verify that every element of PrimRec is quasi total and homogeneous.

One can verify that every element of HFuncs$\,\mathbb{N}$ which is primitive recursive is also quasi total.

One can verify that every function from tuples on $\mathbb{N}$ which is primitive recursive is also length total and there exists an element of PrimRec which is non empty.

6. Examples

Let f be a homogeneous binary relation. We say that f is nullary if and only if:

(Def. 24) $\mathrm{arity}\, f = 0$.

We say that f is unary if and only if:

(Def. 25) $\mathrm{arity}\, f = 1$.

We say that f is binary if and only if:

(Def. 26) $\mathrm{arity}\, f = 2$.

We say that f is ternary if and only if:

(Def. 27) $\mathrm{arity}\, f = 3$.

One can verify the following observations:

* every homogeneous function which is unary is also non empty,
* every homogeneous function which is binary is also non empty, and
* every homogeneous function which is ternary is also non empty.

One can verify the following observations:

* $\mathrm{proj}_1(1)$ is primitive recursive,
* $\mathrm{proj}_2(1)$ is primitive recursive,
* $\mathrm{proj}_2(2)$ is primitive recursive,
* $\mathrm{succ}_1(1)$ is primitive recursive, and
* $\mathrm{succ}_3(3)$ is primitive recursive.

Let i be a natural number. One can verify the following observations:

* $\mathrm{const}_0(i)$ is nullary,
* $\mathrm{const}_1(i)$ is unary,
* $\mathrm{const}_2(i)$ is binary,
* $\mathrm{const}_3(i)$ is ternary,
* $\mathrm{proj}_1(i)$ is unary,
* $\mathrm{proj}_2(i)$ is binary,
* $\mathrm{proj}_3(i)$ is ternary,
* $\mathrm{succ}_1(i)$ is unary,
* $\mathrm{succ}_2(i)$ is binary, and
* $\mathrm{succ}_3(i)$ is ternary.

Let j be a natural number. Note that $\mathrm{const}_i(j)$ is primitive recursive.

One can check the following observations:

* there exists a homogeneous function which is nullary, primitive recursive, and non empty,
* there exists a homogeneous function which is unary and primitive recursive,
* there exists a homogeneous function which is binary and primitive recursive, and
* there exists a homogeneous function which is ternary and primitive recursive.

One can verify the following observations:

* there exists a homogeneous function from tuples on $\mathbb{N}$ which is non empty, nullary, length total, and into $\mathbb{N}$,
* there exists a homogeneous function from tuples on $\mathbb{N}$ which is non empty, unary, length total, and into $\mathbb{N}$,
* there exists a homogeneous function from tuples on $\mathbb{N}$ which is non empty, binary, length total, and into $\mathbb{N}$, and
* there exists a homogeneous function from tuples on $\mathbb{N}$ which is non empty, ternary, length total, and into $\mathbb{N}$.

Let f be a nullary non empty primitive recursive function and let g be a binary primitive recursive function. Observe that $\mathrm{primrec}(f,g,1)$ is primitive recursive and unary.

Let f be a unary primitive recursive function and let g be a ternary primitive recursive function. One can check that $\mathrm{primrec}(f,g,1)$ is primitive recursive and binary and $\mathrm{primrec}(f,g,2)$ is primitive recursive and binary.

We now state four propositions:

(83) Let f_1 be a unary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$ and f_2 be a non empty homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$. Then $(\mathrm{primrec}(f_1,f_2,2))(\langle i, 0\rangle) = f_1(\langle i\rangle)$.

(84) If f_1 is length total and $\mathrm{arity}\, f_1 = 0$, then $(\mathrm{primrec}(f_1,f_2,1))(\langle 0\rangle) = f_1(\emptyset)$.

(85) Let f_1 be a unary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$ and f_2 be a ternary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$. Then $(\mathrm{primrec}(f_1,f_2,2))(\langle i, j+1\rangle) = f_2(\langle i, j, (\mathrm{primrec}(f_1,f_2,2))(\langle i, j\rangle)\rangle)$.

(86) If f_1 is length total and f_2 is length total and $\mathrm{arity}\, f_1 = 0$ and $\mathrm{arity}\, f_2 = 2$, then $(\mathrm{primrec}(f_1,f_2,1))(\langle i+1\rangle) = f_2(\langle i, (\mathrm{primrec}(f_1,f_2,1))(\langle i\rangle)\rangle)$.

Let g be a function. The functor ${}^{\langle 1,?,2\rangle}g$ yields a function and is defined as follows:

(Def. 28) ${}^{\langle 1,?,2\rangle}g = g \cdot \prod^{*}\langle \mathrm{proj}_3(1), \mathrm{proj}_3(3)\rangle$.

Let g be a function into $\mathbb{N}$ and from tuples on $\mathbb{N}$. One can check that ${}^{\langle 1,?,2\rangle}g$ is into $\mathbb{N}$ and from tuples on $\mathbb{N}$.

Let g be a homogeneous function. Note that ${}^{\langle 1,?,2\rangle}g$ is homogeneous.

Let g be a binary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$. Observe that ${}^{\langle 1,?,2\rangle}g$ is non empty, ternary, and length total.

The following propositions are true:

(87) Let f be a binary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$. Then $({}^{\langle 1,?,2\rangle}f)(\langle i, j, k\rangle) = f(\langle i, k\rangle)$.

(88) For every binary primitive recursive function g holds ${}^{\langle 1,?,2\rangle}g \in \mathrm{PrimRec}$.

Let f be a binary primitive recursive homogeneous function. Note that ${}^{\langle 1,?,2\rangle}f$ is primitive recursive and ternary.

The binary primitive recursive function $[+]$ is defined by:

(Def. 29) $[+] = \mathrm{primrec}(\mathrm{proj}_1(1), \mathrm{succ}_3(3), 2)$.

Next we state the proposition

(89) $[+](\langle i, j\rangle) = i + j$.

The binary primitive recursive function $[*]$ is defined by:

(Def. 30) $[*] = \mathrm{primrec}(\mathrm{const}_1(0), {}^{\langle 1,?,2\rangle}[+], 2)$.

One can prove the following proposition

(90) For all natural numbers i, j holds $[*](\langle i, j\rangle) = i \cdot j$.

Let g, h be binary primitive recursive homogeneous functions. Note that $\langle g, h\rangle$ has the same arity.

Let f, g, h be binary primitive recursive functions. Note that $f \cdot \prod^{*}\langle g, h\rangle$ is primitive recursive.

Let f, g, h be binary primitive recursive functions. Observe that $f \cdot \prod^{*}\langle g, h\rangle$ is binary.

Let f be a unary primitive recursive function and let g be a primitive recursive function. One can check that $f \cdot \prod^{*}\langle g\rangle$ is primitive recursive.

Let f be a unary primitive recursive function and let g be a binary primitive recursive function. Observe that $f \cdot \prod^{*}\langle g\rangle$ is binary.

The unary primitive recursive function $[!]$ is defined by:

(Def. 31) $[!] = \mathrm{primrec}(\mathrm{const}_0(1), [*] \cdot \prod^{*}\langle \mathrm{succ}_1(1) \cdot \prod^{*}\langle \mathrm{proj}_2(1)\rangle, \mathrm{proj}_2(2)\rangle, 1)$.

In this article we present several logical schemes. The scheme *Primrec1* deals with a unary length total homogeneous function $\mathcal{A}$ from tuples on $\mathbb{N}$ into $\mathbb{N}$, a binary length total homogeneous function $\mathcal{B}$ from tuples on $\mathbb{N}$ into $\mathbb{N}$, a unary functor $\mathcal{F}$ yielding a natural number, and a binary functor $\mathcal{G}$ yielding a natural number, and states that:

For all natural numbers i, j holds $(\mathcal{A} \cdot \prod^{*}\langle \mathcal{B}\rangle)(\langle i, j\rangle) = \mathcal{F}(\mathcal{G}(i, j))$

provided the parameters meet the following conditions:

- For every natural number i holds $\mathcal{A}(\langle i\rangle) = \mathcal{F}(i)$, and
- For all natural numbers i, j holds $\mathcal{B}(\langle i, j\rangle) = \mathcal{G}(i, j)$.

The scheme *Primrec2* deals with binary length total homogeneous functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ from tuples on $\mathbb{N}$ into $\mathbb{N}$ and three binary functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ yielding natural numbers, and states that:

For all natural numbers i, j holds $(\mathcal{A} \cdot \prod^{*}\langle \mathcal{B}, \mathcal{C}\rangle)(\langle i, j\rangle) = \mathcal{F}(\mathcal{G}(i, j), \mathcal{H}(i, j))$

provided the parameters meet the following conditions:

- For all natural numbers i, j holds $\mathcal{A}(\langle i, j\rangle) = \mathcal{F}(i, j)$,
- For all natural numbers i, j holds $\mathcal{B}(\langle i, j\rangle) = \mathcal{G}(i, j)$, and
- For all natural numbers i, j holds $\mathcal{C}(\langle i, j\rangle) = \mathcal{H}(i, j)$.

The following proposition is true

(91) $[!](\langle i\rangle) = i!$.

The binary primitive recursive function $[\hat{}]$ is defined by:

(Def. 32) $[\hat{}] = \mathrm{primrec}(\mathrm{const}_1(1), {}^{\langle 1,?,2\rangle}[*], 2)$.

One can prove the following proposition

(92) $[\hat{}](\langle i, j\rangle) = i^j$.

The unary primitive recursive function [pred] is defined by:

(Def. 33) $[\mathrm{pred}] = \mathrm{primrec}(\mathrm{const}_0(0), \mathrm{proj}_2(1), 1)$.

Next we state the proposition

(93) $[\mathrm{pred}](\langle 0\rangle) = 0$ and $[\mathrm{pred}](\langle i+1\rangle) = i$.

The binary primitive recursive function $[-]$ is defined by:

(Def. 34) $[-] = \mathrm{primrec}(\mathrm{proj}_1(1), {}^{\langle 1,?,2\rangle}([\mathrm{pred}] \cdot \prod^* \langle \mathrm{proj}_2(2)\rangle), 2)$.

The following proposition is true

(94) $[-](\langle i, j\rangle) = i -' j$.

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