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The Field of Complex Numbers

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Summary. This article contains the definition and many facts about the field of complex numbers.

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The articles [9], [12], [1], [10], [5], [6], [8], [11], [7], [4], [3], and [2] provide the notation and terminology for this paper.

One can prove the following proposition

(2)[1] For all elements x_1, y_1, x_2, y_2 of $\mathbb{R}$ holds $(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$.

The strict double loop structure $\mathbb{C}_{\rm F}$ is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $\mathbb{C}_{\rm F} = \mathbb{C}$,

(ii) the addition of $\mathbb{C}_{\rm F} = +_{\mathbb{C}}$,

(iii) the multiplication of $\mathbb{C}_{\rm F} = \cdot_{\mathbb{C}}$,

(iv) the unity of $\mathbb{C}_{\rm F} = 1_{\mathbb{C}}$, and

(v) the zero of $\mathbb{C}_{\rm F} = 0_{\mathbb{C}}$.

Let us mention that $\mathbb{C}_{\rm F}$ is non empty.

Let us mention that every element of $\mathbb{C}_{\rm F}$ is complex.

One can verify that $\mathbb{C}_{\rm F}$ is add-associative, right zeroed, right complementable, Abelian, commutative, associative, left unital, right unital, distributive, field-like, and non degenerated.

Next we state several propositions:

(3) For all elements x_1, y_1 of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ holds $x_1 + y_1 = x_2 + y_2$.

(4) For every element x_1 of $\mathbb{C}_{\rm F}$ and for every element x_2 of $\mathbb{C}$ such that $x_1 = x_2$ holds $-x_1 = -x_2$.

(5) For all elements x_1, y_1 of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ holds $x_1 - y_1 = x_2 - y_2$.

(6) For all elements x_1, y_1 of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ holds $x_1 \cdot y_1 = x_2 \cdot y_2$.

(7) For every element x_1 of $\mathbb{C}_{\rm F}$ and for every element x_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $x_1 \neq 0_{\mathbb{C}_{\rm F}}$ holds $x_1{}^{-1} = x_2{}^{-1}$.

[1] The proposition (1) has been removed.

(8) For all elements x_1, y_1 of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ and $y_1 \neq 0_{\mathbb{C}_{\rm F}}$ holds $\frac{x_1}{y_1} = \frac{x_2}{y_2}$.

(9) $0_{\mathbb{C}_{\rm F}} = 0_{\mathbb{C}}$.

(10) $\mathbf{1}_{\mathbb{C}_{\rm F}} = 1_{\mathbb{C}}$.

(11) $\mathbf{1}_{\mathbb{C}_{\rm F}} + \mathbf{1}_{\mathbb{C}_{\rm F}} \neq 0_{\mathbb{C}_{\rm F}}$.

Let z be an element of $\mathbb{C}_{\rm F}$. The functor $\overline{z}$ yielding an element of $\mathbb{C}_{\rm F}$ is defined as follows:

(Def. 2) There exists an element z' of $\mathbb{C}$ such that $z = z'$ and $\overline{z} = \overline{z'}$.

Let z be an element of $\mathbb{C}_{\rm F}$. The functor $|z|$ yields a real number and is defined as follows:

(Def. 3) There exists an element z' of $\mathbb{C}$ such that $z = z'$ and $|z| = |z'|$.

Let z be an element of $\mathbb{C}_{\rm F}$. Then $|z|$ is an element of $\mathbb{R}$.
The following proposition is true

(12) For every element x_1 of $\mathbb{C}_{\rm F}$ and for every element x_2 of $\mathbb{C}$ such that $x_1 = x_2$ holds $\overline{x_1} = \overline{x_2}$.

In the sequel z, z_1, z_2, z_3, z_4 are elements of $\mathbb{C}_{\rm F}$.
Next we state a number of propositions:

(29)[2] $-z = (-\mathbf{1}_{\mathbb{C}_{\rm F}}) \cdot z$.

(35)[3] $z_1 - -z_2 = z_1 + z_2$.

(41)[4] $z_1 = (z_1 + z) - z$.

(42) $z_1 = (z_1 - z) + z$.

(47)[5] If $z_1 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_2 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_1{}^{-1} = z_2{}^{-1}$, then $z_1 = z_2$.

(48) If $z_2 \neq 0_{\mathbb{C}_{\rm F}}$ and if $z_1 \cdot z_2 =$ the unity of $\mathbb{C}_{\rm F}$ or $z_2 \cdot z_1 =$ the unity of $\mathbb{C}_{\rm F}$, then $z_1 = z_2{}^{-1}$.

(49) If $z_2 \neq 0_{\mathbb{C}_{\rm F}}$ and if $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = z_3 \cdot z_2{}^{-1}$ and $z_1 = z_2{}^{-1} \cdot z_3$.

(50) (The unity of $\mathbb{C}_{\rm F})^{-1} =$ the unity of $\mathbb{C}_{\rm F}$.

(51) If $z_1 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_2 \neq 0_{\mathbb{C}_{\rm F}}$, then $(z_1 \cdot z_2)^{-1} = z_1{}^{-1} \cdot z_2{}^{-1}$.

(53)[6] If $z \neq 0_{\mathbb{C}_{\rm F}}$, then $(-z)^{-1} = -z^{-1}$.

(55)[7] If $z_1 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_2 \neq 0_{\mathbb{C}_{\rm F}}$, then $z_1{}^{-1} + z_2{}^{-1} = (z_1 + z_2) \cdot (z_1 \cdot z_2)^{-1}$.

(56) If $z_1 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_2 \neq 0_{\mathbb{C}_{\rm F}}$, then $z_1{}^{-1} - z_2{}^{-1} = (z_2 - z_1) \cdot (z_1 \cdot z_2)^{-1}$.

(58)[8] If $z \neq 0_{\mathbb{C}_{\rm F}}$, then $z^{-1} = \frac{\text{the unity of } \mathbb{C}_{\rm F}}{z}$.

(59) $\frac{z}{\text{the unity of } \mathbb{C}_{\rm F}} = z$.

(60) If $z \neq 0_{\mathbb{C}_{\rm F}}$, then $\frac{z}{z} =$ the unity of $\mathbb{C}_{\rm F}$.

(61) If $z \neq 0_{\mathbb{C}_{\rm F}}$, then $\frac{0_{\mathbb{C}_{\rm F}}}{z} = 0_{\mathbb{C}_{\rm F}}$.

(62) If $z_2 \neq 0_{\mathbb{C}_{\rm F}}$ and $\frac{z_1}{z_2} = 0_{\mathbb{C}_{\rm F}}$, then $z_1 = 0_{\mathbb{C}_{\rm F}}$.

[2] The propositions (13)–(28) have been removed.
[3] The propositions (30)–(34) have been removed.
[4] The propositions (36)–(40) have been removed.
[5] The propositions (43)–(46) have been removed.
[6] The proposition (52) has been removed.
[7] The proposition (54) has been removed.
[8] The proposition (57) has been removed.

(63) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_4 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} = \frac{z_1 \cdot z_3}{z_2 \cdot z_4}$.

(64) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $z \cdot \frac{z_1}{z_2} = \frac{z \cdot z_1}{z_2}$.

(65) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $\frac{z_1}{z_2} =$ the unity of $\mathbb{C}_\mathrm{F}$, then $z_1 = z_2$.

(66) If $z \neq 0_{\mathbb{C}_\mathrm{F}}$, then $z_1 = \frac{z_1 \cdot z}{z}$.

(67) If $z_1 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $(\frac{z_1}{z_2})^{-1} = \frac{z_2}{z_1}$.

(68) If $z_1 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{{z_1}^{-1}}{{z_2}^{-1}} = \frac{z_2}{z_1}$.

(69) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1}{{z_2}^{-1}} = z_1 \cdot z_2$.

(70) If $z_1 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{{z_1}^{-1}}{z_2} = (z_1 \cdot z_2)^{-1}$.

(71) If $z_1 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$, then ${z_1}^{-1} \cdot \frac{z}{z_2} = \frac{z}{z_1 \cdot z_2}$.

(72) If $z \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1}{z_2} = \frac{z_1 \cdot z}{z_2 \cdot z}$ and $\frac{z_1}{z_2} = \frac{z \cdot z_1}{z \cdot z_2}$.

(73) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_3 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1}{z_2 \cdot z_3} = \frac{\frac{z_1}{z_2}}{z_3}$.

(74) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_3 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1 \cdot z_3}{z_2} = \frac{z_1}{\frac{z_2}{z_3}}$.

(75) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_3 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_4 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{\frac{z_1}{z_2}}{\frac{z_3}{z_4}} = \frac{z_1 \cdot z_4}{z_2 \cdot z_3}$.

(76) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_4 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1}{z_2} + \frac{z_3}{z_4} = \frac{z_1 \cdot z_4 + z_3 \cdot z_2}{z_2 \cdot z_4}$.

(77) If $z \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1}{z} + \frac{z_2}{z} = \frac{z_1 + z_2}{z}$.

(78) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $-\frac{z_1}{z_2} = \frac{-z_1}{z_2}$ and $-\frac{z_1}{z_2} = \frac{z_1}{-z_2}$.

(79) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1}{z_2} = \frac{-z_1}{-z_2}$.

(80) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$ and $z_4 \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1}{z_2} - \frac{z_3}{z_4} = \frac{z_1 \cdot z_4 - z_3 \cdot z_2}{z_2 \cdot z_4}$.

(81) If $z \neq 0_{\mathbb{C}_\mathrm{F}}$, then $\frac{z_1}{z} - \frac{z_2}{z} = \frac{z_1 - z_2}{z}$.

(82) If $z_2 \neq 0_{\mathbb{C}_\mathrm{F}}$ and if $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = \frac{z_3}{z_2}$.

(83) $\overline{\text{the zero of } \mathbb{C}_\mathrm{F}} =$ the zero of $\mathbb{C}_\mathrm{F}$.

(84) If $\overline{z} =$ the zero of $\mathbb{C}_\mathrm{F}$, then $z =$ the zero of $\mathbb{C}_\mathrm{F}$.

(85) $\overline{\text{the unity of } \mathbb{C}_\mathrm{F}} =$ the unity of $\mathbb{C}_\mathrm{F}$.

(86) $\overline{\overline{z}} = z$.

(87) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(88) $\overline{-z} = -\overline{z}$.

(89) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

(90) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

(91) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\overline{z^{-1}} = \overline{z}^{-1}$.

(92) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$.

(93) $|\text{the zero of } \mathbb{C}_{\mathrm{F}}| = 0$.

(94) If $|z| = 0$, then $z = 0_{\mathbb{C}_{\mathrm{F}}}$.

(95) $0 \leq |z|$.

(96) $z \neq 0_{\mathbb{C}_{\mathrm{F}}}$ iff $0 < |z|$.

(97) $|\text{the unity of } \mathbb{C}_{\mathrm{F}}| = 1$.

(98) $|-z| = |z|$.

(99) $|\overline{z}| = |z|$.

(100) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(101) $|z_1 - z_2| \leq |z_1| + |z_2|$.

(102) $|z_1| - |z_2| \leq |z_1 + z_2|$.

(103) $|z_1| - |z_2| \leq |z_1 - z_2|$.

(104) $|z_1 - z_2| = |z_2 - z_1|$.

(105) $|z_1 - z_2| = 0$ iff $z_1 = z_2$.

(106) $z_1 \neq z_2$ iff $0 < |z_1 - z_2|$.

(107) $|z_1 - z_2| \leq |z_1 - z| + |z - z_2|$.

(108) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

(109) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

(110) If $z \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $|z^{-1}| = |z|^{-1}$.

(111) If $z_2 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $\frac{|z_1|}{|z_2|} = |\frac{z_1}{z_2}|$.

(112) $|z \cdot z| = |z \cdot \overline{z}|$.

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