JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
Inst. of Computer Science, Univ. of Białystok

Inner Products and Angles of Complex Numbers

Wenpai Chang
Shinshu University
Nagano

Yatsuka Nakamura
Shinshu University
Nagano

Piotr Rudnicki
University of Alberta
Edmonton

Summary. An inner product of complex numbers is defined and used to characterize the (counter-clockwise) angle between (a,0) and (0,b) in the complex plane. For complex a, b and c we then define the (counter-clockwise) angle between (a,c) and (c, b) and prove theorems about the sum of internal and external angles of a triangle.

MML Identifier: `COMPLEX2`.

WWW: `http://mizar.org/JFM/Vol15/complex2.html`

The articles [10], [15], [12], [14], [16], [5], [9], [17], [7], [8], [1], [11], [3], [13], [4], [2], and [6] provide the notation and terminology for this paper.

1. PRELIMINARIES

The following propositions are true:

(1) For all real numbers a, b holds $-(a+bi) = -a+(-b)i$.

(2) For all real numbers a, b such that $b > 0$ there exists a real number r such that $r = b \cdot -\lfloor \frac{a}{b} \rfloor + a$ and $0 \leq r$ and $r < b$.

(3) Let a, b, c be real numbers. Suppose $a > 0$ and $b \geq 0$ and $c \geq 0$ and $b < a$ and $c < a$. Let i be an integer. If $b = c + a \cdot i$, then $b = c$.

(4) For all real numbers a, b holds $\sin(a-b) = \sin a \cdot \cos b - \cos a \cdot \sin b$ and $\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b$.

(5) For every real number a holds $\sin(a-\pi) = -\sin(a)$ and $\cos(a-\pi) = -\cos(a)$.

(6) For every real number a holds $\sin(a-\pi) = -\sin a$ and $\cos(a-\pi) = -\cos a$.

(7) For all real numbers a, b such that $a \in]0, \frac{\pi}{2}[$ and $b \in]0, \frac{\pi}{2}[$ holds $a < b$ iff $\sin a < \sin b$.

(8) For all real numbers a, b such that $a \in]\frac{\pi}{2}, \pi[$ and $b \in]\frac{\pi}{2}, \pi[$ holds $a < b$ iff $\sin a > \sin b$.

(9) For every real number a and for every integer i holds $\sin a = \sin(2 \cdot \pi \cdot i + a)$.

(10) For every real number a and for every integer i holds $\cos a = \cos(2 \cdot \pi \cdot i + a)$.

(11) For every real number a such that $\sin a = 0$ holds $\cos a \neq 0$.

(12) For all real numbers a, b such that $0 \leq a$ and $a < 2 \cdot \pi$ and $0 \leq b$ and $b < 2 \cdot \pi$ and $\sin a = \sin b$ and $\cos a = \cos b$ holds $a = b$.

2. More on the Argument of a Complex Number

Let us note that $\mathbb{C}_{\mathrm{F}}$ is non empty.

Let z be an element of $\mathbb{C}$. The functor $\mathrm{Ftize}(z)$ yielding an element of $\mathbb{C}_{\mathrm{F}}$ is defined by:

(Def. 1) $\mathrm{Ftize}(z) = z$.

We now state four propositions:

(13) For every element z of $\mathbb{C}$ holds $\Re(z) = \Re(\mathrm{Ftize}(z))$ and $\Im(z) = \Im(\mathrm{Ftize}(z))$.

(14) For all elements x, y of $\mathbb{C}$ holds $\mathrm{Ftize}(x+y) = \mathrm{Ftize}(x) + \mathrm{Ftize}(y)$.

(15) For every element z of $\mathbb{C}$ holds $z = 0_{\mathbb{C}}$ iff $\mathrm{Ftize}(z) = 0_{\mathbb{C}_{\mathrm{F}}}$.

(16) For every element z of $\mathbb{C}$ holds $|z| = |\mathrm{Ftize}(z)|$.

Let z be an element of $\mathbb{C}$. The functor $\mathrm{Arg}\, z$ yields a real number and is defined by:

(Def. 2) $\mathrm{Arg}\, z = \mathrm{Arg}\, \mathrm{Ftize}(z)$.

Next we state a number of propositions:

(17) For every element z of $\mathbb{C}$ and for every element u of $\mathbb{C}_{\mathrm{F}}$ such that $z = u$ holds $\mathrm{Arg}\, z = \mathrm{Arg}\, u$.

(18) For every element z of $\mathbb{C}$ holds $0 \leq \mathrm{Arg}\, z$ and $\mathrm{Arg}\, z < 2 \cdot \pi$.

(19) For every element z of $\mathbb{C}$ holds $z = |z| \cdot \cos \mathrm{Arg}\, z + (|z| \cdot \sin \mathrm{Arg}\, z)i$.

(20) $\mathrm{Arg}(0_{\mathbb{C}}) = 0$.

(21) Let z be an element of $\mathbb{C}$ and r be a real number. If $z \neq 0$ and $z = |z| \cdot \cos r + (|z| \cdot \sin r)i$ and $0 \leq r$ and $r < 2 \cdot \pi$, then $r = \mathrm{Arg}\, z$.

(22) For every element z of $\mathbb{C}$ such that $z \neq 0_{\mathbb{C}}$ holds if $\mathrm{Arg}\, z < \pi$, then $\mathrm{Arg}(-z) = \mathrm{Arg}\, z + \pi$ and if $\mathrm{Arg}\, z \geq \pi$, then $\mathrm{Arg}(-z) = \mathrm{Arg}\, z - \pi$.

(23) For every real number r such that $r \geq 0$ holds $\mathrm{Arg}(r + 0i) = 0$.

(24) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = 0$ iff $z = |z| + 0i$.

(25) For every element z of $\mathbb{C}$ such that $z \neq 0_{\mathbb{C}}$ holds $\mathrm{Arg}\, z < \pi$ iff $\mathrm{Arg}(-z) \geq \pi$.

(26) For all elements x, y of $\mathbb{C}$ such that $x \neq y$ or $x - y \neq 0_{\mathbb{C}}$ holds $\mathrm{Arg}(x-y) < \pi$ iff $\mathrm{Arg}(y-x) \geq \pi$.

(27) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z \in]0, \pi[$ iff $\Im(z) > 0$.

(28) For every element z of $\mathbb{C}$ such that $\mathrm{Arg}\, z \neq 0$ holds $\mathrm{Arg}\, z < \pi$ iff $\sin \mathrm{Arg}\, z > 0$.

(29) For all elements x, y of $\mathbb{C}$ such that $\mathrm{Arg}\, x < \pi$ and $\mathrm{Arg}\, y < \pi$ holds $\mathrm{Arg}(x+y) < \pi$.

(30) For every real number x such that $x > 0$ holds $\mathrm{Arg}(0 + xi) = \frac{\pi}{2}$.

(31) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z \in]0, \frac{\pi}{2}[$ iff $\Re(z) > 0$ and $\Im(z) > 0$.

(32) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z \in]\frac{\pi}{2}, \pi[$ iff $\Re(z) < 0$ and $\Im(z) > 0$.

(33) For every element z of $\mathbb{C}$ such that $\Im(z) > 0$ holds $\sin \mathrm{Arg}\, z > 0$.

(34) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = 0$ iff $\Re(z) \geq 0$ and $\Im(z) = 0$.

(35) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = \pi$ iff $\Re(z) < 0$ and $\Im(z) = 0$.

(36) For every element z of $\mathbb{C}$ holds $\Im(z) = 0$ iff $\mathrm{Arg}\, z = 0$ or $\mathrm{Arg}\, z = \pi$.

(37) For every element z of $\mathbb{C}$ such that $\operatorname{Arg} z \leq \pi$ holds $\Im(z) \geq 0$.

(38) For every element z of $\mathbb{C}$ such that $z \neq 0$ holds $\cos \operatorname{Arg}(-z) = -\cos \operatorname{Arg} z$ and $\sin \operatorname{Arg}(-z) = -\sin \operatorname{Arg} z$.

(39) For every element a of $\mathbb{C}$ such that $a \neq 0$ holds $\cos \operatorname{Arg} a = \frac{\Re(a)}{|a|}$ and $\sin \operatorname{Arg} a = \frac{\Im(a)}{|a|}$.

(40) For every element a of $\mathbb{C}$ and for every real number r such that $r > 0$ holds $\operatorname{Arg}(a \cdot (r + 0i)) = \operatorname{Arg} a$.

(41) For every element a of $\mathbb{C}$ and for every real number r such that $r < 0$ holds $\operatorname{Arg}(a \cdot (r + 0i)) = \operatorname{Arg}(-a)$.

3. Inner Product

Let x, y be elements of $\mathbb{C}$. The functor $(x|y)$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 3) $(x|y) = x \cdot \overline{y}$.

In the sequel a, b, c, d, x, y, z denote elements of $\mathbb{C}$.
The following propositions are true:

(42) $(x|y) = (\Re(x) \cdot \Re(y) + \Im(x) \cdot \Im(y)) + (-\Re(x) \cdot \Im(y) + \Im(x) \cdot \Re(y))i$.

(43) $(z|z) = (\Re(z) \cdot \Re(z) + \Im(z) \cdot \Im(z)) + 0i$ and $(z|z) = (\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}) + 0i$.

(44) $(z|z) = |z|^{\mathbf{2}} + 0i$ and $|z|^{\mathbf{2}} = \Re((z|z))$.

(45) $|(x|y)| = |x| \cdot |y|$.

(46) If $(x|x) = 0$, then $x = 0$.

(47) $(y|x) = \overline{(x|y)}$.

(48) $((x+y)|z) = (x|z) + (y|z)$.

(49) $(x|(y+z)) = (x|y) + (x|z)$.

(50) $((a \cdot x)|y) = a \cdot (x|y)$.

(51) $(x|(a \cdot y)) = \overline{a} \cdot (x|y)$.

(52) $((a \cdot x)|y) = (x|(\overline{a} \cdot y))$.

(53) $((a \cdot x + b \cdot y)|z) = a \cdot (x|z) + b \cdot (y|z)$.

(54) $(x|(a \cdot y + b \cdot z)) = \overline{a} \cdot (x|y) + \overline{b} \cdot (x|z)$.

(55) $((-x)|y) = (x|-y)$.

(56) $((-x)|y) = -(x|y)$.

(57) $-(x|y) = (x|-y)$.

(58) $((-x)|-y) = (x|y)$.

(59) $((x-y)|z) = (x|z) - (y|z)$.

(60) $(x|(y-z)) = (x|y) - (x|z)$.

(61) $(0_{\mathbb{C}}|x) = 0_{\mathbb{C}}$ and $(x|0_{\mathbb{C}}) = 0_{\mathbb{C}}$.

(62) $((x+y)|(x+y)) = (x|x) + (x|y) + (y|x) + (y|y)$.

(63) $((x-y)|(x-y)) = ((x|x) - (x|y) - (y|x)) + (y|y)$.

(64) $\Re((x|y)) = 0$ iff $\Im((x|y)) = |x| \cdot |y|$ or $\Im((x|y)) = -|x| \cdot |y|$.

4. Rotation

Let a be an element of $\mathbb{C}$ and let r be a real number. The functor $a \circlearrowleft r$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 4) $a \circlearrowleft r = |a| \cdot \cos(r + \mathrm{Arg}\, a) + (|a| \cdot \sin(r + \mathrm{Arg}\, a))i$.

In the sequel r is a real number.

We now state a number of propositions:

(65) $a \circlearrowleft 0 = a$.

(66) $a \circlearrowleft r = 0_{\mathbb{C}}$ iff $a = 0_{\mathbb{C}}$.

(67) $|a \circlearrowleft r| = |a|$.

(68) If $a \neq 0_{\mathbb{C}}$, then there exists an integer i such that $\mathrm{Arg}(a \circlearrowleft r) = 2 \cdot \pi \cdot i + (r + \mathrm{Arg}\, a)$.

(69) $a \circlearrowleft -\mathrm{Arg}\, a = |a| + 0i$.

(70) $\Re(a \circlearrowleft r) = \Re(a) \cdot \cos r - \Im(a) \cdot \sin r$ and $\Im(a \circlearrowleft r) = \Re(a) \cdot \sin r + \Im(a) \cdot \cos r$.

(71) $a + b \circlearrowleft r = (a \circlearrowleft r) + (b \circlearrowleft r)$.

(72) $-a \circlearrowleft r = -(a \circlearrowleft r)$.

(73) $a - b \circlearrowleft r = (a \circlearrowleft r) - (b \circlearrowleft r)$.

(74) $a \circlearrowleft \pi = -a$.

5. Angles

Let a, b be elements of $\mathbb{C}$. The functor $\measuredangle(a,b)$ yielding a real number is defined as follows:

(Def. 5) $\measuredangle(a,b) = \begin{cases} \mathrm{Arg}(b \circlearrowleft -\mathrm{Arg}\, a), & \text{if } \mathrm{Arg}\, a = 0 \text{ or } b \neq 0, \\ 2 \cdot \pi - \mathrm{Arg}\, a, & \text{otherwise.} \end{cases}$

One can prove the following propositions:

(75) If $r \geq 0$, then $\measuredangle(r + 0i, a) = \mathrm{Arg}\, a$.

(76) If $\mathrm{Arg}\, a = \mathrm{Arg}\, b$ and $a \neq 0$ and $b \neq 0$, then $\mathrm{Arg}(a \circlearrowleft r) = \mathrm{Arg}(b \circlearrowleft r)$.

(77) If $r > 0$, then $\measuredangle(a,b) = \measuredangle(a \cdot (r + 0i), b \cdot (r + 0i))$.

(78) If $a \neq 0$ and $b \neq 0$ and $\mathrm{Arg}\, a = \mathrm{Arg}\, b$, then $\mathrm{Arg}(-a) = \mathrm{Arg}(-b)$.

(79) If $a \neq 0$ and $b \neq 0$, then $\measuredangle(a,b) = \measuredangle(a \circlearrowleft r, b \circlearrowleft r)$.

(80) If $r < 0$ and $a \neq 0$ and $b \neq 0$, then $\measuredangle(a,b) = \measuredangle(a \cdot (r + 0i), b \cdot (r + 0i))$.

(81) If $a \neq 0$ and $b \neq 0$, then $\measuredangle(a,b) = \measuredangle(-a, -b)$.

(82) If $b \neq 0$ and $\measuredangle(a,b) = 0$, then $\measuredangle(a, -b) = \pi$.

(83) If $a \neq 0$ and $b \neq 0$, then $\cos \measuredangle(a,b) = \frac{\Re((a|b))}{|a| \cdot |b|}$ and $\sin \measuredangle(a,b) = -\frac{\Im((a|b))}{|a| \cdot |b|}$.

Let x, y, z be elements of $\mathbb{C}$. The functor $\measuredangle(x,y,z)$ yields a real number and is defined as follows:

(Def. 6) $\measuredangle(x,y,z) = \begin{cases} \mathrm{Arg}(z - y) - \mathrm{Arg}(x - y), & \text{if } \mathrm{Arg}(z - y) - \mathrm{Arg}(x - y) \geq 0, \\ 2 \cdot \pi + (\mathrm{Arg}(z - y) - \mathrm{Arg}(x - y)), & \text{otherwise.} \end{cases}$

Next we state a number of propositions:

(84) $0 \leq \measuredangle(x,y,z)$ and $\measuredangle(x,y,z) < 2 \cdot \pi$.

(85) $\measuredangle(x,y,z) = \measuredangle(x-y,0_{\mathbb{C}},z-y)$.

(86) $\measuredangle(a,b,c) = \measuredangle(a+d,b+d,c+d)$.

(87) $\measuredangle(a,b) = \measuredangle(a,0_{\mathbb{C}},b)$.

(88) If $\measuredangle(x,y,z) = 0$, then $\mathrm{Arg}(x-y) = \mathrm{Arg}(z-y)$ and $\measuredangle(z,y,x) = 0$.

(89) If $a \neq 0_{\mathbb{C}}$ and $b \neq 0_{\mathbb{C}}$, then $\Re((a|b)) = 0$ iff $\measuredangle(a,0_{\mathbb{C}},b) = \frac{\pi}{2}$ or $\measuredangle(a,0_{\mathbb{C}},b) = \frac{3}{2}\cdot\pi$.

(90) If $a \neq 0_{\mathbb{C}}$ and $b \neq 0_{\mathbb{C}}$, then $\Im((a|b)) = |a|\cdot|b|$ or $\Im((a|b)) = -|a|\cdot|b|$ iff $\measuredangle(a,0_{\mathbb{C}},b) = \frac{\pi}{2}$ or $\measuredangle(a,0_{\mathbb{C}},b) = \frac{3}{2}\cdot\pi$.

(91) If $x \neq y$ and if $z \neq y$ and if $\measuredangle(x,y,z) = \frac{\pi}{2}$ or $\measuredangle(x,y,z) = \frac{3}{2}\cdot\pi$, then $|x-y|^{\mathbf{2}} + |z-y|^{\mathbf{2}} = |x-z|^{\mathbf{2}}$.

(92) If $a \neq b$ and $b \neq c$, then $\measuredangle(a,b,c) = \measuredangle(a \circlearrowleft r, b \circlearrowleft r, c \circlearrowleft r)$.

(93) $\measuredangle(a,b,a) = 0$.

(94) $\measuredangle(a,b,c) \neq 0$ iff $\measuredangle(a,b,c) + \measuredangle(c,b,a) = 2\cdot\pi$.

(95) If $\measuredangle(a,b,c) \neq 0$, then $\measuredangle(c,b,a) \neq 0$.

(96) If $\measuredangle(a,b,c) = \pi$, then $\measuredangle(c,b,a) = \pi$.

(97) If $a \neq b$ and $a \neq c$ and $b \neq c$, then $\measuredangle(a,b,c) \neq 0$ or $\measuredangle(b,c,a) \neq 0$ or $\measuredangle(c,a,b) \neq 0$.

(98) If $a \neq b$ and $b \neq c$ and $0 < \measuredangle(a,b,c)$ and $\measuredangle(a,b,c) < \pi$, then $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = \pi$ and $0 < \measuredangle(b,c,a)$ and $0 < \measuredangle(c,a,b)$.

(99) If $a \neq b$ and $b \neq c$ and $\measuredangle(a,b,c) > \pi$, then $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = 5\cdot\pi$ and $\measuredangle(b,c,a) > \pi$ and $\measuredangle(c,a,b) > \pi$.

(100) If $a \neq b$ and $b \neq c$ and $\measuredangle(a,b,c) = \pi$, then $\measuredangle(b,c,a) = 0$ and $\measuredangle(c,a,b) = 0$.

(101) If $a \neq b$ and $a \neq c$ and $b \neq c$ and $\measuredangle(a,b,c) = 0$, then $\measuredangle(b,c,a) = 0$ and $\measuredangle(c,a,b) = \pi$ or $\measuredangle(b,c,a) = \pi$ and $\measuredangle(c,a,b) = 0$.

(102) $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = \pi$ or $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = 5\cdot\pi$ iff $a \neq b$ and $a \neq c$ and $b \neq c$.

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Received May 29, 2003

Published January 2, 2004