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Comma Category

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Summary. Comma category of two functors is introduced.

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The articles [8], [4], [10], [9], [11], [1], [2], [5], [3], [6], and [7] provide the notation and terminology for this paper.

Let x be a set. The functor $x_{\mathbf{1},\mathbf{1}}$ yields a set and is defined by:

(Def. 1) $x_{\mathbf{1},\mathbf{1}} = (x_{\mathbf{1}})_{\mathbf{1}}$.

The functor $x_{\mathbf{1},\mathbf{2}}$ yields a set and is defined as follows:

(Def. 2) $x_{\mathbf{1},\mathbf{2}} = (x_{\mathbf{1}})_{\mathbf{2}}$.

The functor $x_{\mathbf{2},\mathbf{1}}$ yielding a set is defined by:

(Def. 3) $x_{\mathbf{2},\mathbf{1}} = (x_{\mathbf{2}})_{\mathbf{1}}$.

The functor $x_{\mathbf{2},\mathbf{2}}$ yielding a set is defined by:

(Def. 4) $x_{\mathbf{2},\mathbf{2}} = (x_{\mathbf{2}})_{\mathbf{2}}$.

In the sequel x, x_1, x_2, y, y_1, y_2 denote sets.

The following proposition is true

(1) $\langle\langle x_1, x_2\rangle, y\rangle_{\mathbf{1},\mathbf{1}} = x_1$ and $\langle\langle x_1, x_2\rangle, y\rangle_{\mathbf{1},\mathbf{2}} = x_2$ and $\langle x, \langle y_1, y_2\rangle\rangle_{\mathbf{2},\mathbf{1}} = y_1$ and $\langle x, \langle y_1, y_2\rangle\rangle_{\mathbf{2},\mathbf{2}} = y_2$.

Let D_1, D_2, D_3 be non empty sets and let x be an element of $[: [: D_1, D_2 :], D_3 :]$. Then $x_{\mathbf{1},\mathbf{1}}$ is an element of D_1. Then $x_{\mathbf{1},\mathbf{2}}$ is an element of D_2.

Let D_1, D_2, D_3 be non empty sets and let x be an element of $[: D_1, [: D_2, D_3 :] :]$. Then $x_{\mathbf{2},\mathbf{1}}$ is an element of D_2. Then $x_{\mathbf{2},\mathbf{2}}$ is an element of D_3.

For simplicity, we adopt the following convention: C, D, E are categories, c, c_1 are objects of C, d, d_1 are objects of D, x is a set, f, f_1 are morphisms of E, g is a morphism of C, h is a morphism of D, F is a functor from C to E, and G is a functor from D to E.

Let us consider C, D, E, let F be a functor from C to E, and let G be a functor from D to E. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. The functor $\mathrm{Obj}_{(F,G)}$ yields a non empty subset of $[: [:$the objects of $C,$ the objects of $D :]$, the morphisms of $E :]$ and is defined by:

(Def. 5) $\mathrm{Obj}_{(F,G)} = \{\langle\langle c, d\rangle, f\rangle : f \in \hom(F(c), G(d))\}$.

In the sequel o, o_1, o_2 are elements of $\mathrm{Obj}_{(F,G)}$.

The following proposition is true

(2) If there exist c, d, f such that $f \in \hom(F(c), G(d))$, then $o = \langle\langle o_{\mathbf{1},\mathbf{1}}, o_{\mathbf{1},\mathbf{2}}\rangle, o_{\mathbf{2}}\rangle$ and $o_{\mathbf{2}} \in \hom(F(o_{\mathbf{1},\mathbf{1}}), G(o_{\mathbf{1},\mathbf{2}}))$ and $\mathrm{dom}(o_{\mathbf{2}}) = F(o_{\mathbf{1},\mathbf{1}})$ and $\mathrm{cod}(o_{\mathbf{2}}) = G(o_{\mathbf{1},\mathbf{2}})$.

Let us consider C, D, E, F, G. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. The functor $\mathrm{Morph}_{(F,G)}$ yields a non empty subset of $[\![:[\![:\mathrm{Obj}_{(F,G)}, \mathrm{Obj}_{(F,G)}:]\!]$, $[\![:$the morphisms of C, the morphisms of $D:]\!]:]\!]$ and is defined by:

(Def. 6) $\mathrm{Morph}_{(F,G)} = \{\langle\langle o_1, o_2\rangle, \langle g, h\rangle\rangle : \mathrm{dom}\, g = (o_1)_{\mathbf{1},\mathbf{1}} \wedge \mathrm{cod}\, g = (o_2)_{\mathbf{1},\mathbf{1}} \wedge \mathrm{dom}\, h = (o_1)_{\mathbf{1},\mathbf{2}} \wedge \mathrm{cod}\, h = (o_2)_{\mathbf{1},\mathbf{2}} \wedge (o_2)_{\mathbf{2}} \cdot F(g) = G(h) \cdot (o_1)_{\mathbf{2}}\}$.

In the sequel k, k_1, k_2, k' denote elements of $\mathrm{Morph}_{(F,G)}$.

Let us consider C, D, E, F, G, k. Then $k_{\mathbf{1},\mathbf{1}}$ is an element of $\mathrm{Obj}_{(F,G)}$. Then $k_{\mathbf{1},\mathbf{2}}$ is an element of $\mathrm{Obj}_{(F,G)}$.

Next we state the proposition

(3) Given c, d, f such that $f \in \hom(F(c), G(d))$. Then $k = \langle\langle k_{\mathbf{1},\mathbf{1}}, k_{\mathbf{1},\mathbf{2}}\rangle, \langle k_{\mathbf{2},\mathbf{1}}, k_{\mathbf{2},\mathbf{2}}\rangle\rangle$ and $\mathrm{dom}(k_{\mathbf{2},\mathbf{1}}) = (k_{\mathbf{1},\mathbf{1}})_{\mathbf{1},\mathbf{1}}$ and $\mathrm{cod}(k_{\mathbf{2},\mathbf{1}}) = (k_{\mathbf{1},\mathbf{2}})_{\mathbf{1},\mathbf{1}}$ and $\mathrm{dom}(k_{\mathbf{2},\mathbf{2}}) = (k_{\mathbf{1},\mathbf{1}})_{\mathbf{1},\mathbf{2}}$ and $\mathrm{cod}(k_{\mathbf{2},\mathbf{2}}) = (k_{\mathbf{1},\mathbf{2}})_{\mathbf{1},\mathbf{2}}$ and $(k_{\mathbf{1},\mathbf{2}})_{\mathbf{2}} \cdot F(k_{\mathbf{2},\mathbf{1}}) = G(k_{\mathbf{2},\mathbf{2}}) \cdot (k_{\mathbf{1},\mathbf{1}})_{\mathbf{2}}$.

Let us consider C, D, E, F, G, k_1, k_2. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. Let us assume that $(k_1)_{\mathbf{1},\mathbf{2}} = (k_2)_{\mathbf{1},\mathbf{1}}$. The functor $k_2 \cdot k_1$ yielding an element of $\mathrm{Morph}_{(F,G)}$ is defined as follows:

(Def. 7) $k_2 \cdot k_1 = \langle\langle (k_1)_{\mathbf{1},\mathbf{1}}, (k_2)_{\mathbf{1},\mathbf{2}}\rangle, \langle (k_2)_{\mathbf{2},\mathbf{1}} \cdot (k_1)_{\mathbf{2},\mathbf{1}}, (k_2)_{\mathbf{2},\mathbf{2}} \cdot (k_1)_{\mathbf{2},\mathbf{2}}\rangle\rangle$.

Let us consider C, D, E, F, G. The functor $\circ_{(F,G)}$ yielding a partial function from $[\![:\mathrm{Morph}_{(F,G)}, \mathrm{Morph}_{(F,G)}:]\!]$ to $\mathrm{Morph}_{(F,G)}$ is defined as follows:

(Def. 8) $\mathrm{dom}(\circ_{(F,G)}) = \{\langle k_1, k_2\rangle : (k_1)_{\mathbf{1},\mathbf{1}} = (k_2)_{\mathbf{1},\mathbf{2}}\}$ and for all k, k' such that $\langle k, k'\rangle \in \mathrm{dom}(\circ_{(F,G)})$ holds $\circ_{(F,G)}(\langle k, k'\rangle) = k \cdot k'$.

Let us consider C, D, E, F, G. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. The functor (F, G) yields a strict category and is defined by the conditions (Def. 9).

(Def. 9)(i) The objects of $(F, G) = \mathrm{Obj}_{(F,G)}$,

(ii) the morphisms of $(F, G) = \mathrm{Morph}_{(F,G)}$,

(iii) for every k holds (the dom-map of $(F, G))(k) = k_{\mathbf{1},\mathbf{1}}$,

(iv) for every k holds (the cod-map of $(F, G))(k) = k_{\mathbf{1},\mathbf{2}}$,

(v) for every o holds (the id-map of $(F, G))(o) = \langle\langle o, o\rangle, \langle \mathrm{id}_{o_{\mathbf{1},\mathbf{1}}}, \mathrm{id}_{o_{\mathbf{1},\mathbf{2}}}\rangle\rangle$, and

(vi) the composition of $(F, G) = \circ_{(F,G)}$.

One can prove the following two propositions:

(4) The objects of $\dot{\circlearrowleft}(x, y) = \{x\}$ and the morphisms of $\dot{\circlearrowleft}(x, y) = \{y\}$.

(5) For all objects a, b of $\dot{\circlearrowleft}(x, y)$ holds $\hom(a, b) = \{y\}$.

Let us consider C, c. The functor $\dot{\circlearrowleft}(c)$ yielding a strict subcategory of C is defined by:

(Def. 10) $\dot{\circlearrowleft}(c) = \dot{\circlearrowleft}(c, \mathrm{id}_c)$.

Let us consider C, c. The functor (c, C) yields a strict category and is defined by:

(Def. 11) $(c, C) = (\overset{\dot{\circlearrowleft}(c)}{\hookrightarrow}, \mathrm{id}_C)$.

The functor (C, c) yielding a strict category is defined by:

(Def. 12) $(C, c) = (\mathrm{id}_C, \overset{\dot{\circlearrowleft}(c)}{\hookrightarrow})$.

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