JOURNAL OF FORMALIZED MATHEMATICS
Volume 7, Released 1995, Published 2003
Inst. of Computer Science, Univ. of Białystok

Continuous, Stable, and Linear Maps of Coherence Spaces

Grzegorz Bancerek
Institute of Mathematics
Polish Academy of Sciences

MML Identifier: COHSP_1.

WWW: http://mizar.org/JFM/Vol7/cohsp_1.html

The articles [14], [13], [9], [17], [15], [18], [6], [5], [10], [16], [4], [11], [1], [2], [12], [3], [8], and [7] provide the notation and terminology for this paper.

1. Directed Sets

One can verify that there exists a coherent space which is finite. Let X be a set. Let us observe that X is binary complete if and only if:

(Def. 1) For every set A such that for all sets a, b such that $a \in A$ and $b \in A$ holds $a \cup b \in X$ holds $\bigcup A \in X$.

Let X be a set. The functor $\mathrm{FlatCoh}(X)$ yields a set and is defined as follows:

(Def. 2) $\mathrm{FlatCoh}(X) = \mathrm{CohSp}(\mathrm{id}_X)$.

The functor $\mathrm{SubFin}(X)$ yielding a subset of X is defined by:

(Def. 3) For every set x holds $x \in \mathrm{SubFin}(X)$ iff $x \in X$ and x is finite.

Next we state three propositions:

(1) For all sets X, x holds $x \in \mathrm{FlatCoh}(X)$ iff $x = \emptyset$ or there exists a set y such that $x = \{y\}$ and $y \in X$.

(2) For every set X holds $\bigcup \mathrm{FlatCoh}(X) = X$.

(3) For every finite subset-closed set X holds $\mathrm{SubFin}(X) = X$.

One can verify that $\{\emptyset\}$ is subset-closed and binary complete. Let X be a set. Observe that 2^X is subset-closed and binary complete and $\mathrm{FlatCoh}(X)$ is non empty, subset-closed, and binary complete.

Let C be a non empty subset-closed set. One can check that $\mathrm{SubFin}(C)$ is non empty and subset-closed.

One can prove the following proposition

(4) $\mathrm{Web}(\{\emptyset\}) = \emptyset$.

The scheme *MinimalElement wrt Incl* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a set a such that $a \in \mathcal{B}$ and $\mathcal{P}[a]$ and for every set b such that $b \in \mathcal{B}$ and $\mathcal{P}[b]$ and $b \subseteq a$ holds $b = a$

provided the following requirements are met:

- $\mathcal{A} \in \mathcal{B}$,
- $\mathcal{P}[\mathcal{A}]$, and
- $\mathcal{A}$ is finite.

Let C be a coherent space. One can check that there exists an element of C which is finite.

Let X be a set. We say that X is $\cup$-directed if and only if:

(Def. 4) For every finite subset Y of X there exists a set a such that $\bigcup Y \subseteq a$ and $a \in X$.

We say that X is $\cap$-directed if and only if:

(Def. 5) For every finite subset Y of X there exists a set a such that for every set y such that $y \in Y$ holds $a \subseteq y$ and $a \in X$.

Let us mention that every set which is $\cup$-directed is also non empty and every set which is $\cap$-directed is also non empty.

One can prove the following propositions:

(5) Let X be a set. Suppose X is $\cup$-directed. Let a, b be sets. If $a \in X$ and $b \in X$, then there exists a set c such that $a \cup b \subseteq c$ and $c \in X$.

(6) Let X be a non empty set. Suppose that for all sets a, b such that $a \in X$ and $b \in X$ there exists a set c such that $a \cup b \subseteq c$ and $c \in X$. Then X is $\cup$-directed.

(7) Let X be a set. Suppose X is $\cap$-directed. Let a, b be sets. If $a \in X$ and $b \in X$, then there exists a set c such that $c \subseteq a \cap b$ and $c \in X$.

(8) Let X be a non empty set. Suppose that for all sets a, b such that $a \in X$ and $b \in X$ there exists a set c such that $c \subseteq a \cap b$ and $c \in X$. Then X is $\cap$-directed.

(9) For every set x holds $\{x\}$ is $\cup$-directed and $\cap$-directed.

(10) For all sets x, y holds $\{x, y, x \cup y\}$ is $\cup$-directed.

(11) For all sets x, y holds $\{x, y, x \cap y\}$ is $\cap$-directed.

Let us note that there exists a set which is $\cup$-directed, $\cap$-directed, and finite.

Let C be a non empty set. Note that there exists a subset of C which is $\cup$-directed, $\cap$-directed, and finite.

We now state the proposition

(12) For every set X holds $\mathrm{Fin}\,X$ is $\cup$-directed and $\cap$-directed.

Let X be a set. Observe that $\mathrm{Fin}\,X$ is $\cup$-directed and $\cap$-directed.

Let C be a subset-closed non empty set. Observe that there exists a subset of C which is preboolean and non empty.

Let C be a subset-closed non empty set and let a be an element of C. Then $\mathrm{Fin}\,a$ is a preboolean non empty subset of C.

We now state the proposition

(13) Let X be a non empty set and Y be a set. Suppose X is $\cup$-directed and $Y \subseteq \bigcup X$ and Y is finite. Then there exists a set Z such that $Z \in X$ and $Y \subseteq Z$.

Let X be a set. We introduce X is multiplicative as a synonym of X is $\cap$-closed.

Let X be a set. We say that X is closed under directed unions if and only if:

(Def. 7)[1] For every subset A of X such that A is $\cup$-directed holds $\bigcup A \in X$.

Let us observe that every set which is subset-closed is also multiplicative.

One can prove the following proposition

(15)[2] For every coherent space C and for every $\cup$-directed subset A of C holds $\bigcup A \in C$.

Let us mention that every coherent space is closed under directed unions.

One can check that there exists a coherent space which is multiplicative and closed under directed unions.

Let C be a closed under directed unions non empty set and let A be a $\cup$-directed subset of C. Then $\bigcup A$ is an element of C.

Let X, Y be sets. We say that X includes lattice of Y if and only if:

(Def. 8) For all sets a, b such that $a \in Y$ and $b \in Y$ holds $a \cap b \in X$ and $a \cup b \in X$.

Next we state the proposition

(16) For every non empty set X such that X includes lattice of X holds X is $\cup$-directed and $\cap$-directed.

Let X, x, y be sets. We say that X includes lattice of x, y if and only if:

(Def. 9) X includes lattice of $\{x, y\}$.

Next we state the proposition

(17) For all sets X, x, y holds X includes lattice of x, y iff $x \in X$ and $y \in X$ and $x \cap y \in X$ and $x \cup y \in X$.

2. Continuous, Stable, and Linear Functions

Let f be a function. We say that f is preserving arbitrary unions if and only if:

(Def. 10) For every subset A of $\operatorname{dom} f$ such that $\bigcup A \in \operatorname{dom} f$ holds $f(\bigcup A) = \bigcup(f^\circ A)$.

We say that f is preserving directed unions if and only if:

(Def. 11) For every subset A of $\operatorname{dom} f$ such that A is $\cup$-directed and $\bigcup A \in \operatorname{dom} f$ holds $f(\bigcup A) = \bigcup(f^\circ A)$.

Let f be a function. We say that f is $\subseteq$-monotone if and only if:

(Def. 12) For all sets a, b such that $a \in \operatorname{dom} f$ and $b \in \operatorname{dom} f$ and $a \subseteq b$ holds $f(a) \subseteq f(b)$.

We say that f is preserving binary intersections if and only if:

(Def. 13) For all sets a, b such that $\operatorname{dom} f$ includes lattice of a, b holds $f(a \cap b) = f(a) \cap f(b)$.

Let us observe that every function which is preserving directed unions is also $\subseteq$-monotone and every function which is preserving arbitrary unions is also preserving directed unions.

We now state two propositions:

(18) Let f be a function. Suppose f is preserving arbitrary unions. Let x, y be sets. If $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} f$ and $x \cup y \in \operatorname{dom} f$, then $f(x \cup y) = f(x) \cup f(y)$.

(19) For every function f such that f is preserving arbitrary unions holds $f(\emptyset) = \emptyset$.

[1] The definition (Def. 6) has been removed.
[2] The proposition (14) has been removed.

Let C_1, C_2 be coherent spaces. Note that there exists a function from C_1 into C_2 which is preserving arbitrary unions and preserving binary intersections.

Let C be a coherent space. Note that there exists a many sorted set indexed by C which is preserving arbitrary unions and preserving binary intersections.

Let f be a function. We say that f is continuous if and only if:

(Def. 14) $\operatorname{dom} f$ is closed under directed unions and f is preserving directed unions.

Let f be a function. We say that f is stable if and only if:

(Def. 15) $\operatorname{dom} f$ is multiplicative and f is continuous and preserving binary intersections.

Let f be a function. We say that f is linear if and only if:

(Def. 16) f is stable and preserving arbitrary unions.

One can verify the following observations:

* every function which is continuous is also preserving directed unions,
* every function which is stable is also preserving binary intersections and continuous, and
* every function which is linear is also preserving arbitrary unions and stable.

Let X be a closed under directed unions set. Note that every many sorted set indexed by X which is preserving directed unions is also continuous.

Let X be a multiplicative set. Observe that every many sorted set indexed by X which is continuous and preserving binary intersections is also stable.

One can check that every function which is stable and preserving arbitrary unions is also linear.

Note that there exists a function which is linear. Let C be a coherent space. Note that there exists a many sorted set indexed by C which is linear. Let B be a coherent space. One can check that there exists a function from B into C which is linear.

Let f be a continuous function. Observe that $\operatorname{dom} f$ is closed under directed unions.

Let f be a stable function. Note that $\operatorname{dom} f$ is multiplicative.

Next we state several propositions:

(20) For every set X holds $\bigcup \operatorname{Fin} X = X$.

(21) For every continuous function f such that $\operatorname{dom} f$ is subset-closed and for every set a such that $a \in \operatorname{dom} f$ holds $f(a) = \bigcup (f^{\circ} \operatorname{Fin} a)$.

(22) Let f be a function. Suppose $\operatorname{dom} f$ is subset-closed. Then f is continuous if and only if the following conditions are satisfied:

(i) $\operatorname{dom} f$ is closed under directed unions,

(ii) f is $\subseteq$-monotone, and

(iii) for all sets a, y such that $a \in \operatorname{dom} f$ and $y \in f(a)$ there exists a set b such that b is finite and $b \subseteq a$ and $y \in f(b)$.

(23) Let f be a function. Suppose $\operatorname{dom} f$ is subset-closed and closed under directed unions. Then f is stable if and only if the following conditions are satisfied:

(i) f is $\subseteq$-monotone, and

(ii) for all sets a, y such that $a \in \operatorname{dom} f$ and $y \in f(a)$ there exists a set b such that b is finite and $b \subseteq a$ and $y \in f(b)$ and for every set c such that $c \subseteq a$ and $y \in f(c)$ holds $b \subseteq c$.

(24) Let f be a function. Suppose $\operatorname{dom} f$ is subset-closed and closed under directed unions. Then f is linear if and only if the following conditions are satisfied:

(i) f is $\subseteq$-monotone, and

(ii) for all sets a, y such that $a \in \operatorname{dom} f$ and $y \in f(a)$ there exists a set x such that $x \in a$ and $y \in f(\{x\})$ and for every set b such that $b \subseteq a$ and $y \in f(b)$ holds $x \in b$.

3. Graph of Continuous Function

Let f be a function. The functor graph(f) yielding a set is defined as follows:

(Def. 17) For every set x holds $x \in \mathrm{graph}(f)$ iff there exists a finite set y and there exists a set z such that $x = \langle y, z\rangle$ and $y \in \mathrm{dom}\, f$ and $z \in f(y)$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2. Then graph(f) is a subset of $[: C_1, \bigcup C_2 :]$.

Let f be a function. One can check that graph(f) is relation-like.

We now state several propositions:

(25) For every function f and for all sets x, y holds $\langle x, y\rangle \in \mathrm{graph}(f)$ iff x is finite and $x \in \mathrm{dom}\, f$ and $y \in f(x)$.

(26) Let f be a $\subseteq$-monotone function and a, b be sets. Suppose $b \in \mathrm{dom}\, f$ and $a \subseteq b$ and b is finite. Let y be a set. If $\langle a, y\rangle \in \mathrm{graph}(f)$, then $\langle b, y\rangle \in \mathrm{graph}(f)$.

(27) Let C_1, C_2 be coherent spaces, f be a function from C_1 into C_2, a be an element of C_1, and y_1, y_2 be sets. If $\langle a, y_1\rangle \in \mathrm{graph}(f)$ and $\langle a, y_2\rangle \in \mathrm{graph}(f)$, then $\{y_1, y_2\} \in C_2$.

(28) Let C_1, C_2 be coherent spaces, f be a $\subseteq$-monotone function from C_1 into C_2, and a, b be elements of C_1. Suppose $a \cup b \in C_1$. Let y_1, y_2 be sets. If $\langle a, y_1\rangle \in \mathrm{graph}(f)$ and $\langle b, y_2\rangle \in \mathrm{graph}(f)$, then $\{y_1, y_2\} \in C_2$.

(29) For all coherent spaces C_1, C_2 and for all continuous functions f, g from C_1 into C_2 such that $\mathrm{graph}(f) = \mathrm{graph}(g)$ holds $f = g$.

(30) Let C_1, C_2 be coherent spaces and X be a subset of $[: C_1, \bigcup C_2 :]$. Suppose that

(i) for every set x such that $x \in X$ holds $x_{\mathbf{1}}$ is finite,

(ii) for all finite elements a, b of C_1 such that $a \subseteq b$ and for every set y such that $\langle a, y\rangle \in X$ holds $\langle b, y\rangle \in X$, and

(iii) for every finite element a of C_1 and for all sets y_1, y_2 such that $\langle a, y_1\rangle \in X$ and $\langle a, y_2\rangle \in X$ holds $\{y_1, y_2\} \in C_2$.

Then there exists a continuous function f from C_1 into C_2 such that $X = \mathrm{graph}(f)$.

(31) Let C_1, C_2 be coherent spaces, f be a continuous function from C_1 into C_2, and a be an element of C_1. Then $f(a) = (\mathrm{graph}(f))^{\circ}\, \mathrm{Fin}\, a$.

4. Trace of Stable Function

Let f be a function. The functor Trace(f) yielding a set is defined by the condition (Def. 18).

(Def. 18) Let x be a set. Then $x \in \mathrm{Trace}(f)$ if and only if there exist sets a, y such that $x = \langle a, y\rangle$ and $a \in \mathrm{dom}\, f$ and $y \in f(a)$ and for every set b such that $b \in \mathrm{dom}\, f$ and $b \subseteq a$ and $y \in f(b)$ holds $a = b$.

One can prove the following proposition

(32) Let f be a function and a, y be sets. Then $\langle a, y\rangle \in \mathrm{Trace}(f)$ if and only if the following conditions are satisfied:

(i) $a \in \mathrm{dom}\, f$,

(ii) $y \in f(a)$, and

(iii) for every set b such that $b \in \mathrm{dom}\, f$ and $b \subseteq a$ and $y \in f(b)$ holds $a = b$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2. Then Trace(f) is a subset of $[: C_1, \bigcup C_2 :]$.

Let f be a function. One can check that Trace(f) is relation-like.

Next we state a number of propositions:

(33) For every continuous function f such that $\operatorname{dom} f$ is subset-closed holds $\operatorname{Trace}(f) \subseteq \operatorname{graph}(f)$.

(34) Let f be a continuous function. Suppose $\operatorname{dom} f$ is subset-closed. Let a, y be sets. If $\langle a, y\rangle \in \operatorname{Trace}(f)$, then a is finite.

(35) Let C_1, C_2 be coherent spaces, f be a $\subseteq$-monotone function from C_1 into C_2, and a_1, a_2 be sets. Suppose $a_1 \cup a_2 \in C_1$. Let y_1, y_2 be sets. If $\langle a_1, y_1\rangle \in \operatorname{Trace}(f)$ and $\langle a_2, y_2\rangle \in \operatorname{Trace}(f)$, then $\{y_1, y_2\} \in C_2$.

(36) Let C_1, C_2 be coherent spaces, f be a preserving binary intersections function from C_1 into C_2, and a_1, a_2 be sets. If $a_1 \cup a_2 \in C_1$, then for every set y such that $\langle a_1, y\rangle \in \operatorname{Trace}(f)$ and $\langle a_2, y\rangle \in \operatorname{Trace}(f)$ holds $a_1 = a_2$.

(37) Let C_1, C_2 be coherent spaces and f, g be stable functions from C_1 into C_2. If $\operatorname{Trace}(f) \subseteq \operatorname{Trace}(g)$, then for every element a of C_1 holds $f(a) \subseteq g(a)$.

(38) For all coherent spaces C_1, C_2 and for all stable functions f, g from C_1 into C_2 such that $\operatorname{Trace}(f) = \operatorname{Trace}(g)$ holds $f = g$.

(39) Let C_1, C_2 be coherent spaces and X be a subset of $[: C_1, \bigcup C_2 :]$. Suppose that

(i) for every set x such that $x \in X$ holds $x_{\mathbf{1}}$ is finite,

(ii) for all elements a, b of C_1 such that $a \cup b \in C_1$ and for all sets y_1, y_2 such that $\langle a, y_1\rangle \in X$ and $\langle b, y_2\rangle \in X$ holds $\{y_1, y_2\} \in C_2$, and

(iii) for all elements a, b of C_1 such that $a \cup b \in C_1$ and for every set y such that $\langle a, y\rangle \in X$ and $\langle b, y\rangle \in X$ holds $a = b$.

Then there exists a stable function f from C_1 into C_2 such that $X = \operatorname{Trace}(f)$.

(40) Let C_1, C_2 be coherent spaces, f be a stable function from C_1 into C_2, and a be an element of C_1. Then $f(a) = (\operatorname{Trace}(f))^{\circ} \operatorname{Fin} a$.

(41) Let C_1, C_2 be coherent spaces, f be a stable function from C_1 into C_2, a be an element of C_1, and y be a set. Then $y \in f(a)$ if and only if there exists an element b of C_1 such that $\langle b, y\rangle \in \operatorname{Trace}(f)$ and $b \subseteq a$.

(42) For all coherent spaces C_1, C_2 there exists a stable function f from C_1 into C_2 such that $\operatorname{Trace}(f) = \emptyset$.

(43) Let C_1, C_2 be coherent spaces, a be a finite element of C_1, and y be a set. If $y \in \bigcup C_2$, then there exists a stable function f from C_1 into C_2 such that $\operatorname{Trace}(f) = \{\langle a, y\rangle\}$.

(44) Let C_1, C_2 be coherent spaces, a be an element of C_1, y be a set, and f be a stable function from C_1 into C_2. Suppose $\operatorname{Trace}(f) = \{\langle a, y\rangle\}$. Let b be an element of C_1. Then

(i) if $a \subseteq b$, then $f(b) = \{y\}$, and

(ii) if $a \not\subseteq b$, then $f(b) = \emptyset$.

(45) Let C_1, C_2 be coherent spaces, f be a stable function from C_1 into C_2, and X be a subset of $\operatorname{Trace}(f)$. Then there exists a stable function g from C_1 into C_2 such that $\operatorname{Trace}(g) = X$.

(46) Let C_1, C_2 be coherent spaces and A be a set. Suppose that for all sets x, y such that $x \in A$ and $y \in A$ there exists a stable function f from C_1 into C_2 such that $x \cup y = \operatorname{Trace}(f)$. Then there exists a stable function f from C_1 into C_2 such that $\bigcup A = \operatorname{Trace}(f)$.

Let C_1, C_2 be coherent spaces. The functor $\operatorname{StabCoh}(C_1, C_2)$ yields a set and is defined as follows:

(Def. 19) For every set x holds $x \in \operatorname{StabCoh}(C_1, C_2)$ iff there exists a stable function f from C_1 into C_2 such that $x = \operatorname{Trace}(f)$.

Let C_1, C_2 be coherent spaces. One can check that $\mathrm{StabCoh}(C_1,C_2)$ is non empty, subset-closed, and binary complete.

One can prove the following propositions:

(47) For all coherent spaces C_1, C_2 and for every stable function f from C_1 into C_2 holds $\mathrm{Trace}(f) \subseteq [:\mathrm{SubFin}(C_1), \bigcup C_2:]$.

(48) For all coherent spaces C_1, C_2 holds $\bigcup \mathrm{StabCoh}(C_1,C_2) = [:\mathrm{SubFin}(C_1), \bigcup C_2:]$.

(49) Let C_1, C_2 be coherent spaces, a, b be finite elements of C_1, and y_1, y_2 be sets. Then $\langle\langle a, y_1\rangle, \langle b, y_2\rangle\rangle \in \mathrm{Web}(\mathrm{StabCoh}(C_1,C_2))$ if and only if one of the following conditions is satisfied:

(i) $a \cup b \notin C_1$ and $y_1 \in \bigcup C_2$ and $y_2 \in \bigcup C_2$, or

(ii) $\langle y_1, y_2\rangle \in \mathrm{Web}(C_2)$ and if $y_1 = y_2$, then $a = b$.

5. Trace of Linear Function

Next we state the proposition

(50) Let C_1, C_2 be coherent spaces and f be a stable function from C_1 into C_2. Then f is linear if and only if for all sets a, y such that $\langle a, y\rangle \in \mathrm{Trace}(f)$ there exists a set x such that $a = \{x\}$.

Let f be a function. The functor $\mathrm{LinTrace}(f)$ yielding a set is defined as follows:

(Def. 20) For every set x holds $x \in \mathrm{LinTrace}(f)$ iff there exist sets y, z such that $x = \langle y, z\rangle$ and $\langle\{y\}, z\rangle \in \mathrm{Trace}(f)$.

Next we state three propositions:

(51) For every function f and for all sets x, y holds $\langle x, y\rangle \in \mathrm{LinTrace}(f)$ iff $\langle\{x\}, y\rangle \in \mathrm{Trace}(f)$.

(52) For every function f such that $f(\emptyset) = \emptyset$ and for all sets x, y such that $\{x\} \in \mathrm{dom} f$ and $y \in f(\{x\})$ holds $\langle x, y\rangle \in \mathrm{LinTrace}(f)$.

(53) For every function f and for all sets x, y such that $\langle x, y\rangle \in \mathrm{LinTrace}(f)$ holds $\{x\} \in \mathrm{dom} f$ and $y \in f(\{x\})$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2. Then $\mathrm{LinTrace}(f)$ is a subset of $[:\bigcup C_1, \bigcup C_2:]$.

Let f be a function. Note that $\mathrm{LinTrace}(f)$ is relation-like.

Let C_1, C_2 be coherent spaces. The functor $\mathrm{LinCoh}(C_1,C_2)$ yielding a set is defined as follows:

(Def. 21) For every set x holds $x \in \mathrm{LinCoh}(C_1,C_2)$ iff there exists a linear function f from C_1 into C_2 such that $x = \mathrm{LinTrace}(f)$.

One can prove the following propositions:

(54) Let C_1, C_2 be coherent spaces, f be a $\subseteq$-monotone function from C_1 into C_2, and x_1, x_2 be sets. Suppose $\{x_1,x_2\} \in C_1$. Let y_1, y_2 be sets. If $\langle x_1, y_1\rangle \in \mathrm{LinTrace}(f)$ and $\langle x_2, y_2\rangle \in \mathrm{LinTrace}(f)$, then $\{y_1,y_2\} \in C_2$.

(55) Let C_1, C_2 be coherent spaces, f be a preserving binary intersections function from C_1 into C_2, and x_1, x_2 be sets. If $\{x_1,x_2\} \in C_1$, then for every set y such that $\langle x_1, y\rangle \in \mathrm{LinTrace}(f)$ and $\langle x_2, y\rangle \in \mathrm{LinTrace}(f)$ holds $x_1 = x_2$.

(56) For all coherent spaces C_1, C_2 and for all linear functions f, g from C_1 into C_2 such that $\mathrm{LinTrace}(f) = \mathrm{LinTrace}(g)$ holds $f = g$.

(57) Let C_1, C_2 be coherent spaces and X be a subset of $[:\bigcup C_1, \bigcup C_2:]$. Suppose that

(i) for all sets a, b such that $\{a,b\} \in C_1$ and for all sets y_1, y_2 such that $\langle a, y_1\rangle \in X$ and $\langle b, y_2\rangle \in X$ holds $\{y_1,y_2\} \in C_2$, and

(ii) for all sets a, b such that $\{a,b\} \in C_1$ and for every set y such that $\langle a, y\rangle \in X$ and $\langle b, y\rangle \in X$ holds $a = b$.

Then there exists a linear function f from C_1 into C_2 such that $X = \mathrm{LinTrace}(f)$.

(58) Let C_1, C_2 be coherent spaces, f be a linear function from C_1 into C_2, and a be an element of C_1. Then $f(a) = (\mathrm{LinTrace}(f))^{\circ}a$.

(59) For all coherent spaces C_1, C_2 there exists a linear function f from C_1 into C_2 such that $\mathrm{LinTrace}(f) = \emptyset$.

(60) Let C_1, C_2 be coherent spaces, x be a set, and y be a set. Suppose $x \in \bigcup C_1$ and $y \in \bigcup C_2$. Then there exists a linear function f from C_1 into C_2 such that $\mathrm{LinTrace}(f) = \{\langle x, y\rangle\}$.

(61) Let C_1, C_2 be coherent spaces, x be a set, and y be a set. Suppose $x \in \bigcup C_1$ and $y \in \bigcup C_2$. Let f be a linear function from C_1 into C_2. Suppose $\mathrm{LinTrace}(f) = \{\langle x, y\rangle\}$. Let a be an element of C_1. Then

(i) if $x \in a$, then $f(a) = \{y\}$, and

(ii) if $x \notin a$, then $f(a) = \emptyset$.

(62) Let C_1, C_2 be coherent spaces, f be a linear function from C_1 into C_2, and X be a subset of $\mathrm{LinTrace}(f)$. Then there exists a linear function g from C_1 into C_2 such that $\mathrm{LinTrace}(g) = X$.

(63) Let C_1, C_2 be coherent spaces and A be a set. Suppose that for all sets x, y such that $x \in A$ and $y \in A$ there exists a linear function f from C_1 into C_2 such that $x \cup y = \mathrm{LinTrace}(f)$. Then there exists a linear function f from C_1 into C_2 such that $\bigcup A = \mathrm{LinTrace}(f)$.

Let C_1, C_2 be coherent spaces. Note that $\mathrm{LinCoh}(C_1,C_2)$ is non empty, subset-closed, and binary complete.

The following propositions are true:

(64) For all coherent spaces C_1, C_2 holds $\bigcup \mathrm{LinCoh}(C_1,C_2) = [:\bigcup C_1, \bigcup C_2:]$.

(65) Let C_1, C_2 be coherent spaces, x_1, x_2 be sets, and y_1, y_2 be sets. Then $\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle \in \mathrm{Web}(\mathrm{LinCoh}(C_1,C_2))$ if and only if the following conditions are satisfied:

(i) $x_1 \in \bigcup C_1$,

(ii) $x_2 \in \bigcup C_1$, and

(iii) $\langle x_1, x_2\rangle \notin \mathrm{Web}(C_1)$ and $y_1 \in \bigcup C_2$ and $y_2 \in \bigcup C_2$ or $\langle y_1, y_2\rangle \in \mathrm{Web}(C_2)$ and if $y_1 = y_2$, then $x_1 = x_2$.

6. Negation of Coherence Spaces

Let C be a coherent space. The functor $\neg C$ yields a set and is defined by:

(Def. 22) $\neg C = \{a; a \text{ ranges over subsets of } \bigcup C : \bigwedge_{b:\text{element of } C} \bigvee_{x:\text{set}} a \cap b \subseteq \{x\}\}$.

The following proposition is true

(66) Let C be a coherent space and x be a set. Then $x \in \neg C$ if and only if the following conditions are satisfied:

(i) $x \subseteq \bigcup C$, and

(ii) for every element a of C there exists a set z such that $x \cap a \subseteq \{z\}$.

Let C be a coherent space. Observe that $\neg C$ is non empty, subset-closed, and binary complete.

We now state several propositions:

(67) For every coherent space C holds $\bigcup \neg C = \bigcup C$.

(68) For every coherent space C and for all sets x, y such that $x \neq y$ and $\{x, y\} \in C$ holds $\{x, y\} \notin \neg C$.

(69) For every coherent space C and for all sets x, y such that $\{x, y\} \subseteq \bigcup C$ and $\{x, y\} \notin C$ holds $\{x, y\} \in \neg C$.

(70) For every coherent space C and for all sets x, y holds $\langle x, y\rangle \in \mathrm{Web}(\neg C)$ iff $x \in \bigcup C$ but $y \in \bigcup C$ but $x = y$ or $\langle x, y\rangle \notin \mathrm{Web}(C)$.

(71) For every coherent space C holds $\neg\neg C = C$.

(72) $\neg\{\emptyset\} = \{\emptyset\}$.

(73) For every set X holds $\neg \mathrm{FlatCoh}(X) = 2^X$ and $\neg(2^X) = \mathrm{FlatCoh}(X)$.

7. Product and Coproduct on Coherence Spaces

Let x, y be sets. The functor $x \uplus y$ yields a set and is defined as follows:

(Def. 23) $x \uplus y = \bigcup \mathrm{disjoint}\langle x, y\rangle$.

Next we state a number of propositions:

(74) For all sets x, y holds $x \uplus y = [: x, \{1\} :] \cup [: y, \{2\} :]$.

(75) For every set x holds $x \uplus \emptyset = [: x, \{1\} :]$ and $\emptyset \uplus x = [: x, \{2\} :]$.

(76) For all sets x, y, z such that $z \in x \uplus y$ holds $z = \langle z_{\mathbf{1}}, z_{\mathbf{2}}\rangle$ but $z_{\mathbf{2}} = 1$ and $z_{\mathbf{1}} \in x$ or $z_{\mathbf{2}} = 2$ and $z_{\mathbf{1}} \in y$.

(77) For all sets x, y, z holds $\langle z, 1\rangle \in x \uplus y$ iff $z \in x$.

(78) For all sets x, y, z holds $\langle z, 2\rangle \in x \uplus y$ iff $z \in y$.

(79) For all sets x_1, y_1, x_2, y_2 holds $x_1 \uplus y_1 \subseteq x_2 \uplus y_2$ iff $x_1 \subseteq x_2$ and $y_1 \subseteq y_2$.

(80) For all sets x, y, z such that $z \subseteq x \uplus y$ there exist sets x_1, y_1 such that $z = x_1 \uplus y_1$ and $x_1 \subseteq x$ and $y_1 \subseteq y$.

(81) For all sets x_1, y_1, x_2, y_2 holds $x_1 \uplus y_1 = x_2 \uplus y_2$ iff $x_1 = x_2$ and $y_1 = y_2$.

(82) For all sets x_1, y_1, x_2, y_2 holds $(x_1 \uplus y_1) \cup (x_2 \uplus y_2) = x_1 \cup x_2 \uplus y_1 \cup y_2$.

(83) For all sets x_1, y_1, x_2, y_2 holds $(x_1 \uplus y_1) \cap (x_2 \uplus y_2) = x_1 \cap x_2 \uplus y_1 \cap y_2$.

Let C_1, C_2 be coherent spaces. The functor $C_1 \sqcap C_2$ yielding a set is defined as follows:

(Def. 24) $C_1 \sqcap C_2 = \{a \uplus b : a$ ranges over elements of C_1, b ranges over elements of $C_2\}$.

The functor $C_1 \sqcup C_2$ yields a set and is defined by:

(Def. 25) $C_1 \sqcup C_2 = \{a \uplus \emptyset : a$ ranges over elements of $C_1\} \cup \{\emptyset \uplus b : b$ ranges over elements of $C_2\}$.

Next we state several propositions:

(84) Let C_1, C_2 be coherent spaces and x be a set. Then $x \in C_1 \sqcap C_2$ if and only if there exists an element a of C_1 and there exists an element b of C_2 such that $x = a \uplus b$.

(85) For all coherent spaces C_1, C_2 and for all sets x, y holds $x \uplus y \in C_1 \sqcap C_2$ iff $x \in C_1$ and $y \in C_2$.

(86) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \sqcap C_2) = \bigcup C_1 \uplus \bigcup C_2$.

(87) For all coherent spaces C_1, C_2 and for all sets x, y holds $x \uplus y \in C_1 \sqcup C_2$ iff $x \in C_1$ and $y = \emptyset$ or $x = \emptyset$ and $y \in C_2$.

(88) Let C_1, C_2 be coherent spaces and x be a set. Suppose $x \in C_1 \sqcup C_2$. Then there exists an element a of C_1 and there exists an element b of C_2 such that $x = a \uplus b$ but $a = \emptyset$ or $b = \emptyset$.

(89) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \sqcup C_2) = \bigcup C_1 \uplus \bigcup C_2$.

Let C_1, C_2 be coherent spaces. Observe that $C_1 \sqcap C_2$ is non empty, subset-closed, and binary complete and $C_1 \sqcup C_2$ is non empty, subset-closed, and binary complete.

In the sequel C_1, C_2 are coherent spaces.

Next we state several propositions:

(90) For all sets x, y holds $\langle\langle x, 1\rangle, \langle y, 1\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_1)$.

(91) For all sets x, y holds $\langle\langle x, 2\rangle, \langle y, 2\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_2)$.

(92) For all sets x, y such that $x \in \bigcup C_1$ and $y \in \bigcup C_2$ holds $\langle\langle x, 1\rangle, \langle y, 2\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$ and $\langle\langle y, 2\rangle, \langle x, 1\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$.

(93) For all sets x, y holds $\langle\langle x, 1\rangle, \langle y, 1\rangle\rangle \in \mathrm{Web}(C_1 \sqcup C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_1)$.

(94) For all sets x, y holds $\langle\langle x, 2\rangle, \langle y, 2\rangle\rangle \in \mathrm{Web}(C_1 \sqcup C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_2)$.

(95) For all sets x, y holds $\langle\langle x, 1\rangle, \langle y, 2\rangle\rangle \notin \mathrm{Web}(C_1 \sqcup C_2)$ and $\langle\langle y, 2\rangle, \langle x, 1\rangle\rangle \notin \mathrm{Web}(C_1 \sqcup C_2)$.

(96) $\neg(C_1 \sqcap C_2) = \neg C_1 \sqcup \neg C_2$.

Let C_1, C_2 be coherent spaces. The functor $C_1 \otimes C_2$ yielding a set is defined by:

(Def. 26) $C_1 \otimes C_2 = \bigcup\{2^{[: a, b :]} : a$ ranges over elements of C_1, b ranges over elements of $C_2\}$.

One can prove the following proposition

(97) Let C_1, C_2 be coherent spaces and x be a set. Then $x \in C_1 \otimes C_2$ if and only if there exists an element a of C_1 and there exists an element b of C_2 such that $x \subseteq [: a, b :]$.

Let C_1, C_2 be coherent spaces. Note that $C_1 \otimes C_2$ is non empty.

The following proposition is true

(98) For all coherent spaces C_1, C_2 and for every element a of $C_1 \otimes C_2$ holds $\pi_1(a) \in C_1$ and $\pi_2(a) \in C_2$ and $a \subseteq [: \pi_1(a), \pi_2(a) :]$.

Let C_1, C_2 be coherent spaces. Note that $C_1 \otimes C_2$ is subset-closed and binary complete.

Next we state two propositions:

(99) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \otimes C_2) = [: \bigcup C_1, \bigcup C_2 :]$.

(100) For all sets x_1, y_1, x_2, y_2 holds $\langle\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\rangle \in \mathrm{Web}(C_1 \otimes C_2)$ iff $\langle x_1, y_1\rangle \in \mathrm{Web}(C_1)$ and $\langle x_2, y_2\rangle \in \mathrm{Web}(C_2)$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. Curried and uncurried functions. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_5.html`.

[3] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[4] Grzegorz Bancerek. Tarski's classes and ranks. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/classes1.html`.

[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[8] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[9] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[10] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[11] Jarosław Kotowicz and Konrad Raczkowski. Coherent space. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/coh_sp.html`.

[12] Andrzej Nędzusiak. σ-fields and probability. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/prob_1.html`.

[13] Andrzej Trybulec. Enumerated sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/enumset1.html`.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[15] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/mcart_1.html`.

[16] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finsub_1.html`.

[17] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[18] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received August 30, 1995

Published January 2, 2004