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On the Closure Operator and the Closure System of Many Sorted Sets

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Summary. In this paper definitions of many sorted closure system and many sorted closure operator are introduced. These notations are also introduced in [9], but in another meaning. In this article closure system is absolutely multiplicative subset family of many sorted sets and in [9] is many sorted absolutely multiplicative subset family of many sorted sets. Analogously, closure operator is function between many sorted sets and in [9] is many sorted function from a many sorted set into a many sorted set.

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The articles [11], [5], [15], [10], [16], [2], [4], [3], [6], [12], [13], [14], [1], [8], and [7] provide the notation and terminology for this paper.

1. Preliminaries

We adopt the following convention: i, x, I denote sets, A, B, M denote many sorted sets indexed by I, and f, f_1 denote functions.

Next we state two propositions:

(1) For every non empty set M and for all elements X, Y of M such that $X \subseteq Y$ holds $\mathrm{id}_M(X) \subseteq \mathrm{id}_M(Y)$.

(3)[1] Let I be a non empty set, A be a many sorted set indexed by I, and B be a many sorted subset indexed by A. Then $\mathrm{rng}\, B \subseteq \bigcup \mathrm{rng}(2^A)$.

Let us mention that every set which is empty is also functional.

Let us observe that there exists a set which is empty and functional.

Let f, g be functions. One can check that $\{f, g\}$ is functional.

2. Set of Many Sorted Subsets of a Many Sorted Set

Let us consider I, M. The functor $\mathrm{Bool}(M)$ yielding a set is defined by:

(Def. 1) $x \in \mathrm{Bool}(M)$ iff x is a many sorted subset indexed by M.

[1] The proposition (2) has been removed.

Let us consider I, M. Observe that $\mathrm{Bool}(M)$ is non empty and functional and has common domain.

Let us consider I, M. A family of many sorted subsets indexed by M is a subset of $\mathrm{Bool}(M)$.

Let us consider I, M. Then $\mathrm{Bool}(M)$ is a family of many sorted subsets indexed by M.

Let us consider I, M. One can verify that there exists a family of many sorted subsets indexed by M which is non empty and functional and has common domain.

Let us consider I, M. Observe that there exists a family of many sorted subsets indexed by M which is empty and finite.

In the sequel S_1, S_2 denote families of many sorted subsets indexed by M.

Let us consider I, M and let S be a non empty family of many sorted subsets indexed by M. We see that the element of S is a many sorted subset indexed by M.

The following propositions are true:

(4) $S_1 \cup S_2$ is a family of many sorted subsets indexed by M.

(5) $S_1 \cap S_2$ is a family of many sorted subsets indexed by M.

(6) $S_1 \setminus x$ is a family of many sorted subsets indexed by M.

(7) $S_1 \dot{-} S_2$ is a family of many sorted subsets indexed by M.

(8) If $A \subseteq M$, then $\{A\}$ is a family of many sorted subsets indexed by M.

(9) If $A \subseteq M$ and $B \subseteq M$, then $\{A, B\}$ is a family of many sorted subsets indexed by M.

In the sequel E, T denote elements of $\mathrm{Bool}(M)$.

We now state four propositions:

(10) $E \cap T \in \mathrm{Bool}(M)$.

(11) $E \cup T \in \mathrm{Bool}(M)$.

(12) $E \setminus A \in \mathrm{Bool}(M)$.

(13) $E \dot{-} T \in \mathrm{Bool}(M)$.

3. Many Sorted Operator corresponding to the Operator on Many Sorted Subsets

Let S be a functional set. The functor $|S|$ yields a function and is defined as follows:

(Def. 3)[2] (i) There exists a non empty functional set A such that $A = S$ and $\mathrm{dom}\,|S| = \bigcap\{\mathrm{dom}\,x : x$ ranges over elements of $A\}$ and for every i such that $i \in \mathrm{dom}\,|S|$ holds $|S|(i) = \{x(i) : x$ ranges over elements of $A\}$ if $S \neq \emptyset$,

(ii) $|S| = \emptyset$, otherwise.

We now state the proposition

(14) For every non empty family S_1 of many sorted subsets indexed by M holds $\mathrm{dom}\,|S_1| = I$.

Let S be an empty functional set. Observe that $|S|$ is empty.

Let us consider I, M and let S be a family of many sorted subsets indexed by M. The functor $|{:}S{:}|$ yielding a many sorted set indexed by I is defined by:

(Def. 4) $$|{:}S{:}| = \begin{cases} \text{(i)} \quad |S|, \text{ if } S \neq \emptyset, \\ \mathbf{0}_I, \text{ otherwise.} \end{cases}$$

Let us consider I, M and let S be an empty family of many sorted subsets indexed by M. One can verify that $|{:}S{:}|$ is empty yielding.

Next we state the proposition

[2] The definition (Def. 2) has been removed.

(15) If S_1 is non empty, then for every i such that $i \in I$ holds $|{:}S_1{:}|(i) = \{x(i); x$ ranges over elements of $\mathrm{Bool}(M)$: $x \in S_1\}$.

Let us consider I, M and let S_1 be a non empty family of many sorted subsets indexed by M. One can check that $|{:}S_1{:}|$ is non-empty.

We now state several propositions:

(16) $\mathrm{dom}\,|\{f\}| = \mathrm{dom}\,f$.

(17) $\mathrm{dom}\,|\{f, f_1\}| = \mathrm{dom}\,f \cap \mathrm{dom}\,f_1$.

(18) If $i \in \mathrm{dom}\,f$, then $|\{f\}|(i) = \{f(i)\}$.

(19) If $i \in I$ and $S_1 = \{f\}$, then $|{:}S_1{:}|(i) = \{f(i)\}$.

(20) If $i \in \mathrm{dom}\,|\{f, f_1\}|$, then $|\{f, f_1\}|(i) = \{f(i), f_1(i)\}$.

(21) If $i \in I$ and $S_1 = \{f, f_1\}$, then $|{:}S_1{:}|(i) = \{f(i), f_1(i)\}$.

Let us consider I, M, S_1. Then $|{:}S_1{:}|$ is a subset family of M.

One can prove the following propositions:

(22) If $A \in S_1$, then $A \in |{:}S_1{:}|$.

(23) If $S_1 = \{A, B\}$, then $\bigcup|{:}S_1{:}| = A \cup B$.

(24) If $S_1 = \{E, T\}$, then $\bigcap|{:}S_1{:}| = E \cap T$.

(25) Let Z be a many sorted subset indexed by M. Suppose that for every many sorted set Z_1 indexed by I such that $Z_1 \in S_1$ holds $Z \subseteq Z_1$. Then $Z \subseteq \bigcap|{:}S_1{:}|$.

(26) $|{:}\mathrm{Bool}(M){:}| = 2^M$.

Let us consider I, M and let I_1 be a family of many sorted subsets indexed by M. We say that I_1 is additive if and only if:

(Def. 5) For all A, B such that $A \in I_1$ and $B \in I_1$ holds $A \cup B \in I_1$.

We say that I_1 is absolutely-additive if and only if:

(Def. 6) For every family F of many sorted subsets indexed by M such that $F \subseteq I_1$ holds $\bigcup|{:}F{:}| \in I_1$.

We say that I_1 is multiplicative if and only if:

(Def. 7) For all A, B such that $A \in I_1$ and $B \in I_1$ holds $A \cap B \in I_1$.

We say that I_1 is absolutely-multiplicative if and only if:

(Def. 8) For every family F of many sorted subsets indexed by M such that $F \subseteq I_1$ holds $\bigcap|{:}F{:}| \in I_1$.

We say that I_1 is properly upper bound if and only if:

(Def. 9) $M \in I_1$.

We say that I_1 is properly lower bound if and only if:

(Def. 10) $\mathbf{0}_I \in I_1$.

Let us consider I, M. One can check that there exists a family of many sorted subsets indexed by M which is non empty, functional, additive, absolutely-additive, multiplicative, absolutely-multiplicative, properly upper bound, and properly lower bound and has common domain.

Let us consider I, M. Then $\mathrm{Bool}(M)$ is an additive absolutely-additive multiplicative absolutely-multiplicative properly upper bound properly lower bound family of many sorted subsets indexed by M.

Let us consider I, M. Observe that every family of many sorted subsets indexed by M which is absolutely-additive is also additive.

Let us consider I, M. Observe that every family of many sorted subsets indexed by M which is absolutely-multiplicative is also multiplicative.

Let us consider I, M. One can check that every family of many sorted subsets indexed by M which is absolutely-multiplicative is also properly upper bound.

Let us consider I, M. Observe that every family of many sorted subsets indexed by M which is properly upper bound is also non empty.

Let us consider I, M. Observe that every family of many sorted subsets indexed by M which is absolutely-additive is also properly lower bound.

Let us consider I, M. One can verify that every family of many sorted subsets indexed by M which is properly lower bound is also non empty.

4. Properties of Closure Operators

Let us consider I, M. A set operation in M is a function from $\mathrm{Bool}(M)$ into $\mathrm{Bool}(M)$.

Let us consider I, M, let f be a set operation in M, and let x be an element of $\mathrm{Bool}(M)$. Then $f(x)$ is an element of $\mathrm{Bool}(M)$.

Let us consider I, M and let I_1 be a set operation in M. We say that I_1 is reflexive if and only if:

(Def. 12)[3] For every element x of $\mathrm{Bool}(M)$ holds $x \subseteq I_1(x)$.

We say that I_1 is monotonic if and only if:

(Def. 13) For all elements x, y of $\mathrm{Bool}(M)$ such that $x \subseteq y$ holds $I_1(x) \subseteq I_1(y)$.

We say that I_1 is idempotent if and only if:

(Def. 14) For every element x of $\mathrm{Bool}(M)$ holds $I_1(x) = I_1(I_1(x))$.

We say that I_1 is topological if and only if:

(Def. 15) For all elements x, y of $\mathrm{Bool}(M)$ holds $I_1(x \cup y) = I_1(x) \cup I_1(y)$.

Let us consider I, M. Note that there exists a set operation in M which is reflexive, monotonic, idempotent, and topological.

The following propositions are true:

(27) $\mathrm{id}_{\mathrm{Bool}(A)}$ is a reflexive set operation in A.

(28) $\mathrm{id}_{\mathrm{Bool}(A)}$ is a monotonic set operation in A.

(29) $\mathrm{id}_{\mathrm{Bool}(A)}$ is an idempotent set operation in A.

(30) $\mathrm{id}_{\mathrm{Bool}(A)}$ is a topological set operation in A.

In the sequel g, h are set operations in M.

We now state three propositions:

(31) If $E = M$ and g is reflexive, then $E = g(E)$.

(32) If g is reflexive and for every element X of $\mathrm{Bool}(M)$ holds $g(X) \subseteq X$, then g is idempotent.

(33) For every element A of $\mathrm{Bool}(M)$ such that $A = E \cap T$ holds if g is monotonic, then $g(A) \subseteq g(E) \cap g(T)$.

Let us consider I, M. Note that every set operation in M which is topological is also monotonic.

One can prove the following proposition

[3] The definition (Def. 11) has been removed.

(34) For every element A of $\mathrm{Bool}(M)$ such that $A = E \setminus T$ holds if g is topological, then $g(E) \setminus g(T) \subseteq g(A)$.

Let us consider I, M, h, g. Then $g \cdot h$ is a set operation in M.
Next we state four propositions:

(35) If g is reflexive and h is reflexive, then $g \cdot h$ is reflexive.

(36) If g is monotonic and h is monotonic, then $g \cdot h$ is monotonic.

(37) If g is idempotent and h is idempotent and $g \cdot h = h \cdot g$, then $g \cdot h$ is idempotent.

(38) If g is topological and h is topological, then $g \cdot h$ is topological.

5. On the Closure Operator and the Closure System

In the sequel S is a 1-sorted structure.

Let us consider S. We introduce closure system structures over S which are extensions of many-sorted structure over S and are systems

⟨ sorts, a family ⟩,

where the sorts constitute a many sorted set indexed by the carrier of S and the family is a family of many sorted subsets indexed by the sorts.

In the sequel M_1 denotes a many-sorted structure over S.

Let us consider S and let I_1 be a closure system structure over S. We say that I_1 is additive if and only if:

(Def. 16) The family of I_1 is additive.

We say that I_1 is absolutely-additive if and only if:

(Def. 17) The family of I_1 is absolutely-additive.

We say that I_1 is multiplicative if and only if:

(Def. 18) The family of I_1 is multiplicative.

We say that I_1 is absolutely-multiplicative if and only if:

(Def. 19) The family of I_1 is absolutely-multiplicative.

We say that I_1 is properly upper bound if and only if:

(Def. 20) The family of I_1 is properly upper bound.

We say that I_1 is properly lower bound if and only if:

(Def. 21) The family of I_1 is properly lower bound.

Let us consider S, M_1. The functor $\mathrm{Full}(M_1)$ yielding a closure system structure over S is defined as follows:

(Def. 22) $\mathrm{Full}(M_1) = \langle$the sorts of M_1, Bool(the sorts of M_1)$\rangle$.

Let us consider S, M_1. Note that $\mathrm{Full}(M_1)$ is strict, additive, absolutely-additive, multiplicative, absolutely-multiplicative, properly upper bound, and properly lower bound.

Let us consider S and let M_1 be a non-empty many-sorted structure over S. Observe that $\mathrm{Full}(M_1)$ is non-empty.

Let us consider S. Note that there exists a closure system structure over S which is strict, non-empty, additive, absolutely-additive, multiplicative, absolutely-multiplicative, properly upper bound, and properly lower bound.

Let us consider S and let C_1 be an additive closure system structure over S. Note that the family of C_1 is additive.

Let us consider S and let C_1 be an absolutely-additive closure system structure over S. Note that the family of C_1 is absolutely-additive.

Let us consider S and let C_1 be a multiplicative closure system structure over S. Note that the family of C_1 is multiplicative.

Let us consider S and let C_1 be an absolutely-multiplicative closure system structure over S. One can verify that the family of C_1 is absolutely-multiplicative.

Let us consider S and let C_1 be a properly upper bound closure system structure over S. One can check that the family of C_1 is properly upper bound.

Let us consider S and let C_1 be a properly lower bound closure system structure over S. Note that the family of C_1 is properly lower bound.

Let us consider S, let M be a non-empty many sorted set indexed by the carrier of S, and let F be a family of many sorted subsets indexed by M. One can check that $\langle M, F\rangle$ is non-empty.

Let us consider S, M_1 and let F be an additive family of many sorted subsets indexed by the sorts of M_1. Note that $\langle$the sorts of M_1, $F\rangle$ is additive.

Let us consider S, M_1 and let F be an absolutely-additive family of many sorted subsets indexed by the sorts of M_1. Observe that $\langle$the sorts of M_1, $F\rangle$ is absolutely-additive.

Let us consider S, M_1 and let F be a multiplicative family of many sorted subsets indexed by the sorts of M_1. One can check that $\langle$the sorts of M_1, $F\rangle$ is multiplicative.

Let us consider S, M_1 and let F be an absolutely-multiplicative family of many sorted subsets indexed by the sorts of M_1. Observe that $\langle$the sorts of M_1, $F\rangle$ is absolutely-multiplicative.

Let us consider S, M_1 and let F be a properly upper bound family of many sorted subsets indexed by the sorts of M_1. One can check that $\langle$the sorts of M_1, $F\rangle$ is properly upper bound.

Let us consider S, M_1 and let F be a properly lower bound family of many sorted subsets indexed by the sorts of M_1. Observe that $\langle$the sorts of M_1, $F\rangle$ is properly lower bound.

Let us consider S. One can check that every closure system structure over S which is absolutely-additive is also additive.

Let us consider S. Observe that every closure system structure over S which is absolutely-multiplicative is also multiplicative.

Let us consider S. Note that every closure system structure over S which is absolutely-multiplicative is also properly upper bound.

Let us consider S. One can verify that every closure system structure over S which is absolutely-additive is also properly lower bound.

Let us consider S. A closure system of S is an absolutely-multiplicative closure system structure over S.

Let us consider I, M. A closure operator of M is a reflexive monotonic idempotent set operation in M.

Next we state the proposition

(39) Let A be a many sorted set indexed by the carrier of S, f be a reflexive monotonic set operation in A, and D be a family of many sorted subsets indexed by A. Suppose $D = \{x; x$ ranges over elements of $\mathrm{Bool}(A)$: $f(x) = x\}$. Then $\langle A, D\rangle$ is a closure system of S.

Let us consider S, let A be a many sorted set indexed by the carrier of S, and let g be a closure operator of A. The functor $\mathrm{ClSys}(g)$ yields a strict closure system of S and is defined by:

(Def. 23) The sorts of $\mathrm{ClSys}(g) = A$ and the family of $\mathrm{ClSys}(g) = \{x; x$ ranges over elements of $\mathrm{Bool}(A)$: $g(x) = x\}$.

Let us consider S, let A be a closure system of S, and let C be a many sorted subset indexed by the sorts of A. The functor $\overline{C}$ yields an element of Bool(the sorts of A) and is defined by the condition (Def. 24).

(Def. 24) There exists a family F of many sorted subsets indexed by the sorts of A such that $\overline{C} = \bigcap |{:}F{:}|$ and $F = \{X; X$ ranges over elements of Bool(the sorts of A): $C \subseteq X \wedge X \in$ the family of $A\}$.

The following propositions are true:

(40) Let D be a closure system of S, a be an element of Bool(the sorts of D), and f be a set operation in the sorts of D. Suppose $a \in$ the family of D and for every element x of Bool(the sorts of D) holds $f(x) = \overline{x}$. Then $f(a) = a$.

(41) Let D be a closure system of S, a be an element of Bool(the sorts of D), and f be a set operation in the sorts of D. Suppose $f(a) = a$ and for every element x of Bool(the sorts of D) holds $f(x) = \overline{x}$. Then $a \in$ the family of D.

(42) Let D be a closure system of S and f be a set operation in the sorts of D. Suppose that for every element x of Bool(the sorts of D) holds $f(x) = \overline{x}$. Then f is reflexive, monotonic, and idempotent.

Let us consider S and let D be a closure system of S. The functor $\mathrm{ClOp}(D)$ yields a closure operator of the sorts of D and is defined as follows:

(Def. 25) For every element x of Bool(the sorts of D) holds $(\mathrm{ClOp}(D))(x) = \overline{x}$.

Next we state two propositions:

(43) For every many sorted set A indexed by the carrier of S and for every closure operator f of A holds $\mathrm{ClOp}(\mathrm{ClSys}(f)) = f$.

(44) For every closure system D of S holds $\mathrm{ClSys}(\mathrm{ClOp}(D)) =$ the closure system structure of D.

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