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Universal Classes

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Summary. In the article we have shown that there exist universal classes, i.e. there are sets which are closed w.r.t. basic set theory operations.

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The articles [12], [8], [13], [4], [9], [11], [14], [6], [7], [2], [3], [1], [5], and [10] provide the notation and terminology for this paper.

We use the following convention: m is a cardinal number, A, B, C are ordinal numbers, and x, y, X, Y, W are sets.

Let us note that every set which is a Tarski class is also subset-closed.

Let X be a set. Observe that $\mathbf{T}(X)$ is a Tarski class.

We now state four propositions:

(1) If W is subset-closed and $X \in W$, then $X \not\approx W$ and $\overline{\overline{X}} < \overline{\overline{W}}$.

(3)[1] If W is a Tarski class and $x \in W$ and $y \in W$, then $\{x\} \in W$ and $\{x,y\} \in W$.

(4) If W is a Tarski class and $x \in W$ and $y \in W$, then $\langle x, y\rangle \in W$.

(5) If W is a Tarski class and $X \in W$, then $\mathbf{T}(X) \subseteq W$.

The scheme *TC* concerns a unary predicate $\mathcal{P}$, and states that:

For every X holds $\mathcal{P}[\mathbf{T}(X)]$

provided the following requirement is met:

- For every X such that X is a Tarski class holds $\mathcal{P}[X]$.

One can prove the following propositions:

(6) If W is a Tarski class and $A \in W$, then $\mathrm{succ}A \in W$ and $A \subseteq W$.

(7) If $A \in \mathbf{T}(W)$, then $\mathrm{succ}A \in \mathbf{T}(W)$ and $A \subseteq \mathbf{T}(W)$.

(8) If W is subset-closed and X is transitive and $X \in W$, then $X \subseteq W$.

(9) If X is transitive and $X \in \mathbf{T}(W)$, then $X \subseteq \mathbf{T}(W)$.

(10) If W is a Tarski class, then $\mathrm{On}\,W = \overline{\overline{W}}$.

(11) $\mathrm{On}\,\mathbf{T}(W) = \overline{\overline{\mathbf{T}(W)}}$.

(12) If W is a Tarski class and $X \in W$, then $\overline{\overline{X}} \in W$.

[1] The proposition (2) has been removed.

(13) If $X \in \mathbf{T}(W)$, then $\overline{\overline{X}} \in \mathbf{T}(W)$.

(14) If W is a Tarski class and $x \in \overline{\overline{W}}$, then $x \in W$.

(15) If $x \in \overline{\overline{\mathbf{T}(W)}}$, then $x \in \mathbf{T}(W)$.

(16) If W is a Tarski class and $m < \overline{\overline{W}}$, then $m \in W$.

(17) If $m < \overline{\overline{\mathbf{T}(W)}}$, then $m \in \mathbf{T}(W)$.

(18) If W is a Tarski class and $m \in W$, then $m \subseteq W$.

(19) If $m \in \mathbf{T}(W)$, then $m \subseteq \mathbf{T}(W)$.

(20) If W is a Tarski class, then $\overline{\overline{W}}$ is a limit ordinal number.

(21) If W is a Tarski class and $W \neq \emptyset$, then $\overline{\overline{W}} \neq 0$ and $\overline{\overline{W}} \neq \emptyset$ and $\overline{\overline{W}}$ is a limit ordinal number.

(22) $\overline{\overline{\mathbf{T}(W)}} \neq 0$ and $\overline{\overline{\mathbf{T}(W)}} \neq \emptyset$ and $\overline{\overline{\mathbf{T}(W)}}$ is a limit ordinal number.

In the sequel L is a transfinite sequence.
Next we state a number of propositions:

(23) If W is a Tarski class and if $X \in W$ and W is transitive or $X \in W$ and $X \subseteq W$ or $\overline{\overline{X}} < \overline{\overline{W}}$ and $X \subseteq W$, then $W^X \subseteq W$.

(24) If $X \in \mathbf{T}(W)$ and W is transitive or $X \in \mathbf{T}(W)$ and $X \subseteq \mathbf{T}(W)$ or $\overline{\overline{X}} < \overline{\overline{\mathbf{T}(W)}}$ and $X \subseteq \mathbf{T}(W)$, then $\mathbf{T}(W)^X \subseteq \mathbf{T}(W)$.

(25) If $\mathrm{dom}\, L$ is a limit ordinal number and for every A such that $A \in \mathrm{dom}\, L$ holds $L(A) = \mathbf{R}_A$, then $\mathbf{R}_{\mathrm{dom}\, L} = \bigcup L$.

(26) If W is a Tarski class and $A \in \mathrm{On}\, W$, then $\overline{\overline{\mathbf{R}_A}} < \overline{\overline{W}}$ and $\mathbf{R}_A \in W$.

(27) If $A \in \mathrm{On}\, \mathbf{T}(W)$, then $\overline{\overline{\mathbf{R}_A}} < \overline{\overline{\mathbf{T}(W)}}$ and $\mathbf{R}_A \in \mathbf{T}(W)$.

(28) If W is a Tarski class, then $\mathbf{R}_{\overline{\overline{W}}} \subseteq W$.

(29) $\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}} \subseteq \mathbf{T}(W)$.

(30) If W is a Tarski class and transitive and $X \in W$, then $\mathrm{rk}(X) \in W$.

(31) If W is a Tarski class and transitive, then $W \subseteq \mathbf{R}_{\overline{\overline{W}}}$.

(32) If W is a Tarski class and transitive, then $\mathbf{R}_{\overline{\overline{W}}} = W$.

(33) If W is a Tarski class and $A \in \mathrm{On}\, W$, then $\overline{\overline{\mathbf{R}_A}} \leq \overline{\overline{W}}$.

(34) If $A \in \mathrm{On}\, \mathbf{T}(W)$, then $\overline{\overline{\mathbf{R}_A}} \leq \overline{\overline{\mathbf{T}(W)}}$.

(35) If W is a Tarski class, then $\overline{\overline{W}} = \overline{\overline{\mathbf{R}_{\overline{\overline{W}}}}}$.

(36) $\overline{\overline{\mathbf{T}(W)}} = \overline{\overline{\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}}}$.

(37) If W is a Tarski class and $X \subseteq \mathbf{R}_{\overline{\overline{W}}}$, then $X \approx \mathbf{R}_{\overline{\overline{W}}}$ or $X \in \mathbf{R}_{\overline{\overline{W}}}$.

(38) If $X \subseteq \mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$, then $X \approx \mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$ or $X \in \mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$.

(39) If W is a Tarski class, then $\mathbf{R}_{\overline{\overline{W}}}$ is a Tarski class.

(40) $\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$ is a Tarski class.

(41) If X is transitive and $A \in \mathrm{rk}(X)$, then there exists Y such that $Y \in X$ and $\mathrm{rk}(Y) = A$.

(42) If X is transitive, then $\overline{\overline{\mathrm{rk}(X)}} \leq \overline{\overline{X}}$.

(43) If W is a Tarski class and X is transitive and $X \in W$, then $X \in \mathbf{R}_{\overline{\overline{W}}}$.

(44) If X is transitive and $X \in \mathbf{T}(W)$, then $X \in \mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$.

(45) If W is transitive, then $\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$ is a Tarski class of W.

(46) If W is transitive, then $\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}} = \mathbf{T}(W)$.

Let I_1 be a set. We say that I_1 is universal if and only if:

(Def. 1) I_1 is transitive and a Tarski class.

Let us mention that every set which is universal is also transitive and a Tarski class and every set which is transitive and a Tarski class is also universal.

Let us mention that there exists a set which is universal and non empty.

A universal class is a universal non empty set.

In the sequel U_1, U_2, U_3, U_4 are universal classes.

We now state three propositions:

(50)[2] $\mathrm{On}\, U_4$ is an ordinal number.

(51) If X is transitive, then $\mathbf{T}(X)$ is universal.

(52) $\mathbf{T}(U_4)$ is a universal class.

Let us consider U_4. One can check that $\mathrm{On}\, U_4$ is ordinal and $\mathbf{T}(U_4)$ is universal.

We now state the proposition

(53) $\mathbf{T}(A)$ is universal.

Let us consider A. Observe that $\mathbf{T}(A)$ is universal.

The following propositions are true:

(54) $U_4 = \mathbf{R}_{\mathrm{On}\, U_4}$.

(55) $\mathrm{On}\, U_4 \neq \emptyset$ and $\mathrm{On}\, U_4$ is a limit ordinal number.

(56) $U_1 \in U_2$ or $U_1 = U_2$ or $U_2 \in U_1$.

(57) $U_1 \subseteq U_2$ or $U_2 \in U_1$.

(58) U_1 and U_2 are $\subseteq$-comparable.

(59) If $U_1 \in U_2$ and $U_2 \in U_3$, then $U_1 \in U_3$.

(61)[3] $U_1 \cup U_2$ is a universal class and $U_1 \cap U_2$ is a universal class.

(62) $\emptyset \in U_4$.

(63) If $x \in U_4$, then $\{x\} \in U_4$.

(64) If $x \in U_4$ and $y \in U_4$, then $\{x, y\} \in U_4$ and $\langle x, y \rangle \in U_4$.

(65) If $X \in U_4$, then $2^X \in U_4$ and $\bigcup X \in U_4$ and $\bigcap X \in U_4$.

(66) If $X \in U_4$ and $Y \in U_4$, then $X \cup Y \in U_4$ and $X \cap Y \in U_4$ and $X \setminus Y \in U_4$ and $X \dot{-} Y \in U_4$.

[2] The propositions (47)–(49) have been removed.
[3] The proposition (60) has been removed.

(67) If $X \in U_4$ and $Y \in U_4$, then $[:X, Y:] \in U_4$ and $Y^X \in U_4$.

In the sequel u, v denote elements of U_4.

Let us consider U_1. One can verify that there exists an element of U_1 which is non empty.

Let us consider U_4, u. Then $\{u\}$ is an element of U_4. Then 2^u is a non empty element of U_4. Then $\bigcup u$ is an element of U_4. Then $\bigcap u$ is an element of U_4. Let us consider v. Then $\{u, v\}$ is an element of U_4. Then $\langle u, v\rangle$ is an element of U_4. Then $u \cup v$ is an element of U_4. Then $u \cap v$ is an element of U_4. Then $u \setminus v$ is an element of U_4. Then $u \dot{-} v$ is an element of U_4. Then $[:u, v:]$ is an element of U_4. Then v^u is an element of U_4.

The universal class $\mathbf{U}_0$ is defined as follows:

(Def. 2) $\mathbf{U}_0 = \mathbf{T}(\emptyset)$.

One can prove the following three propositions:

(69)[4] $\overline{\overline{\mathbf{R}_\omega}} = \overline{\overline{\omega}}$.

(70) $\mathbf{R}_\omega$ is a Tarski class.

(71) $\mathbf{U}_0 = \mathbf{R}_\omega$.

The universal class $\mathbf{U}_1$ is defined as follows:

(Def. 3) $\mathbf{U}_1 = \mathbf{T}(\mathbf{U}_0)$.

Let X be a set. One can verify that $X^{*\in}$ is transitive.

Let X be a transitive set. Observe that $\mathbf{T}(X)$ is transitive.

Let A be an ordinal number. Observe that $\mathbf{R}_A$ is transitive.

Let X be a set. The functor Universe_closure(X) yielding a universal class is defined by:

(Def. 4) $X \subseteq$ Universe_closure(X) and for every universal class Y such that $X \subseteq Y$ holds Universe_closure(X) $\subseteq Y$.

A set of a finite rank is an element of $\mathbf{U}_0$. A *Set* is an element of $\mathbf{U}_1$. Let us consider A. The functor $\mathbf{U}_A$ is defined by the condition (Def. 5).

(Def. 5) There exists L such that

(i) $\mathbf{U}_A = \operatorname{last} L$,

(ii) $\operatorname{dom} L = \operatorname{succ} A$,

(iii) $L(\emptyset) = \mathbf{U}_0$,

(iv) for every C such that $\operatorname{succ} C \in \operatorname{succ} A$ holds $L(\operatorname{succ} C) = \mathbf{T}(L(C))$, and

(v) for every C such that $C \in \operatorname{succ} A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) =$ Universe_closure($\bigcup(L \restriction C)$).

Let us consider A. One can verify that $\mathbf{U}_A$ is universal and non empty.

One can prove the following propositions:

(75)[5] $\mathbf{U}_\emptyset = \mathbf{U}_0$.

(76) $\mathbf{U}_{\operatorname{succ} A} = \mathbf{T}(\mathbf{U}_A)$.

(77) $\mathbf{U}_{\mathbf{1}} = \mathbf{U}_1$.

(78) If $A \neq \emptyset$ and A is a limit ordinal number and $\operatorname{dom} L = A$ and for every B such that $B \in A$ holds $L(B) = \mathbf{U}_B$, then $\mathbf{U}_A =$ Universe_closure($\bigcup L$).

(79) $\mathbf{U}_0 \subseteq U_4$ and $\mathbf{T}(\emptyset) \subseteq U_4$ and $\mathbf{U}_\emptyset \subseteq U_4$.

(80) $A \in B$ iff $\mathbf{U}_A \in \mathbf{U}_B$.

(81) If $\mathbf{U}_A = \mathbf{U}_B$, then $A = B$.

(82) $A \subseteq B$ iff $\mathbf{U}_A \subseteq \mathbf{U}_B$.

[4] The proposition (68) has been removed.

[5] The propositions (72)–(74) have been removed.

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