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Combining of Circuits[1]

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Summary. We continue the formalisation of circuits started in [11],[12],[10], [13]. Our goal was to work out the notation of combining circuits which could be employed to prove the properties of real circuits.

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The articles [15], [21], [19], [16], [22], [4], [3], [7], [9], [5], [14], [6], [8], [20], [1], [2], [23], [17], [18], [11], [12], [10], and [13] provide the notation and terminology for this paper.

1. Combining of Many Sorted Signatures

Let S be a many sorted signature. A gate of S is an element of the operation symbols of S.

Let A be a set and let f be a function. One can check that $A \longmapsto f$ is function yielding.

Let f, g be non-empty functions. Observe that $f{+}\cdot g$ is non-empty.

Let A, B be sets, let f be a many sorted set indexed by A, and let g be a many sorted set indexed by B. Then $f{+}\cdot g$ is a many sorted set indexed by $A \cup B$.

The following propositions are true:

(1) For all functions f_1, f_2, g_1, g_2 such that $\operatorname{rng} g_1 \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} g_2 \subseteq \operatorname{dom} f_2$ and $f_1 \approx f_2$ holds $(f_1{+}\cdot f_2) \cdot (g_1{+}\cdot g_2) = f_1 \cdot g_1{+}\cdot f_2 \cdot g_2$.

(2) For all functions f_1, f_2, g such that $\operatorname{rng} g \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} g \subseteq \operatorname{dom} f_2$ and $f_1 \approx f_2$ holds $f_1 \cdot g = f_2 \cdot g$.

(3) Let A, B be sets, f be a many sorted set indexed by A, and g be a many sorted set indexed by B. If $f \subseteq g$, then $f^\# \subseteq g^\#$.

(4) For all sets X, Y, x, y holds $X \longmapsto x \approx Y \longmapsto y$ iff $x = y$ or X misses Y.

(5) For all functions f, g, h such that $f \approx g$ and $g \approx h$ and $h \approx f$ holds $f{+}\cdot g \approx h$.

(6) For every set X and for every non empty set Y and for every finite sequence p of elements of X holds $(X \longmapsto Y)^\#(p) = Y^{\operatorname{len} p}$.

Let A be a set, let f_1, g_1 be non-empty many sorted sets indexed by A, let B be a set, let f_2, g_2 be non-empty many sorted sets indexed by B, let h_1 be a many sorted function from f_1 into g_1, and let h_2 be a many sorted function from f_2 into g_2. Then $h_1{+}\cdot h_2$ is a many sorted function from $f_1{+}\cdot f_2$ into $g_1{+}\cdot g_2$.

Let S_1, S_2 be many sorted signatures. The predicate $S_1 \approx S_2$ is defined as follows:

[1]This work was written while the second author visited Shinshu University, July–August 1994.

(Def. 1) The arity of $S_1 \approx$ the arity of S_2 and the result sort of $S_1 \approx$ the result sort of S_2.

Let us notice that the predicate $S_1 \approx S_2$ is reflexive and symmetric.

Let S_1, S_2 be non empty many sorted signatures. The functor $S_1 +\cdot S_2$ yielding a strict non empty many sorted signature is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of $S_1 +\cdot S_2 =$ (the carrier of S_1) $\cup$ (the carrier of S_2),

(ii) the operation symbols of $S_1 +\cdot S_2 =$ (the operation symbols of S_1) $\cup$ (the operation symbols of S_2),

(iii) the arity of $S_1 +\cdot S_2 =$ (the arity of S_1) $+\cdot$ (the arity of S_2), and

(iv) the result sort of $S_1 +\cdot S_2 =$ (the result sort of S_1) $+\cdot$ (the result sort of S_2).

The following propositions are true:

(7) For all non empty many sorted signatures S_1, S_2, S_3 such that $S_1 \approx S_2$ and $S_2 \approx S_3$ and $S_3 \approx S_1$ holds $S_1 +\cdot S_2 \approx S_3$.

(8) For every non empty many sorted signature S holds $S +\cdot S =$ the many sorted signature of S.

(9) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ holds $S_1 +\cdot S_2 = S_2 +\cdot S_1$.

(10) For all non empty many sorted signatures S_1, S_2, S_3 holds $(S_1 +\cdot S_2) +\cdot S_3 = S_1 +\cdot (S_2 +\cdot S_3)$.

Let us note that there exists a function which is one-to-one.

The following propositions are true:

(11) Let f be an one-to-one function and S_1, S_2 be circuit-like non empty many sorted signatures. Suppose the result sort of $S_1 \subseteq f$ and the result sort of $S_2 \subseteq f$. Then $S_1 +\cdot S_2$ is circuit-like.

(12) For all circuit-like non empty many sorted signatures S_1, S_2 such that $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InnerVertices}(S_2)$ holds $S_1 +\cdot S_2$ is circuit-like.

(13) For all non empty many sorted signatures S_1, S_2 such that S_1 is not void or S_2 is not void holds $S_1 +\cdot S_2$ is non void.

(14) For all finite non empty many sorted signatures S_1, S_2 holds $S_1 +\cdot S_2$ is finite.

Let S_1 be a non void non empty many sorted signature and let S_2 be a non empty many sorted signature. One can verify that $S_1 +\cdot S_2$ is non void and $S_2 +\cdot S_1$ is non void.

Next we state several propositions:

(15) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ holds $\mathrm{InnerVertices}(S_1 +\cdot S_2) = \mathrm{InnerVertices}(S_1) \cup \mathrm{InnerVertices}(S_2)$ and $\mathrm{InputVertices}(S_1 +\cdot S_2) \subseteq \mathrm{InputVertices}(S_1) \cup \mathrm{InputVertices}(S_2)$.

(16) For all non empty many sorted signatures S_1, S_2 and for every vertex v_2 of S_2 such that $v_2 \in \mathrm{InputVertices}(S_1 +\cdot S_2)$ holds $v_2 \in \mathrm{InputVertices}(S_2)$.

(17) Let S_1, S_2 be non empty many sorted signatures. If $S_1 \approx S_2$, then for every vertex v_1 of S_1 such that $v_1 \in \mathrm{InputVertices}(S_1 +\cdot S_2)$ holds $v_1 \in \mathrm{InputVertices}(S_1)$.

(18) Let S_1 be a non empty many sorted signature, S_2 be a non void non empty many sorted signature, o_2 be an operation symbol of S_2, and o be an operation symbol of $S_1 +\cdot S_2$. Suppose $o_2 = o$. Then $\mathrm{Arity}(o) = \mathrm{Arity}(o_2)$ and the result sort of $o =$ the result sort of o_2.

(19) Let S_1 be a non empty many sorted signature and S_2, S be circuit-like non void non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$. Let v_2 be a vertex of S_2. Suppose $v_2 \in \mathrm{InnerVertices}(S_2)$. Let v be a vertex of S. If $v_2 = v$, then $v \in \mathrm{InnerVertices}(S)$ and the action at $v =$ the action at v_2.

(20) Let S_1 be a non void non empty many sorted signature and S_2 be a non empty many sorted signature. Suppose $S_1 \approx S_2$. Let o_1 be an operation symbol of S_1 and o be an operation symbol of $S_1 {+\cdot} S_2$. Suppose $o_1 = o$. Then $\mathrm{Arity}(o) = \mathrm{Arity}(o_1)$ and the result sort of $o =$ the result sort of o_1.

(21) Let S_1, S be circuit-like non void non empty many sorted signatures and S_2 be a non empty many sorted signature. Suppose $S_1 \approx S_2$ and $S = S_1 {+\cdot} S_2$. Let v_1 be a vertex of S_1. Suppose $v_1 \in \mathrm{InnerVertices}(S_1)$. Let v be a vertex of S. If $v_1 = v$, then $v \in \mathrm{InnerVertices}(S)$ and the action at $v =$ the action at v_1.

2. Combining of Circuits

Let S_1, S_2 be non empty many sorted signatures, let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. The predicate $A_1 \approx A_2$ is defined as follows:

(Def. 3) $S_1 \approx S_2$ and the sorts of $A_1 \approx$ the sorts of A_2 and the characteristics of $A_1 \approx$ the characteristics of A_2.

Let S_1, S_2 be non empty many sorted signatures, let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Let us assume that the sorts of $A_1 \approx$ the sorts of A_2. The functor $A_1 {+\cdot} A_2$ yields a strict non-empty algebra over $S_1 {+\cdot} S_2$ and is defined by the conditions (Def. 4).

(Def. 4)(i) The sorts of $A_1 {+\cdot} A_2 =$ (the sorts of A_1)$+\cdot$(the sorts of A_2), and

(ii) the characteristics of $A_1 {+\cdot} A_2 =$ (the characteristics of A_1)$+\cdot$(the characteristics of A_2).

The following propositions are true:

(22) For every non void non empty many sorted signature S and for every algebra A over S holds $A \approx A$.

(23) Let S_1, S_2 be non void non empty many sorted signatures, A_1 be an algebra over S_1, and A_2 be an algebra over S_2. If $A_1 \approx A_2$, then $A_2 \approx A_1$.

(24) Let S_1, S_2, S_3 be non empty many sorted signatures, A_1 be a non-empty algebra over S_1, A_2 be a non-empty algebra over S_2, and A_3 be an algebra over S_3. If $A_1 \approx A_2$ and $A_2 \approx A_3$ and $A_3 \approx A_1$, then $A_1 {+\cdot} A_2 \approx A_3$.

(25) Let S be a strict non empty many sorted signature and A be a non-empty algebra over S. Then $A {+\cdot} A =$ the algebra of A.

(26) Let S_1, S_2 be non empty many sorted signatures, A_1 be a non-empty algebra over S_1, and A_2 be a non-empty algebra over S_2. If $A_1 \approx A_2$, then $A_1 {+\cdot} A_2 = A_2 {+\cdot} A_1$.

(27) Let S_1, S_2, S_3 be non empty many sorted signatures, A_1 be a non-empty algebra over S_1, A_2 be a non-empty algebra over S_2, and A_3 be a non-empty algebra over S_3. Suppose that

(i) the sorts of $A_1 \approx$ the sorts of A_2,

(ii) the sorts of $A_2 \approx$ the sorts of A_3, and

(iii) the sorts of $A_3 \approx$ the sorts of A_1.

Then $(A_1 {+\cdot} A_2) {+\cdot} A_3 = A_1 {+\cdot} (A_2 {+\cdot} A_3)$.

(28) Let S_1, S_2 be non empty many sorted signatures, A_1 be a locally-finite non-empty algebra over S_1, and A_2 be a locally-finite non-empty algebra over S_2. If the sorts of $A_1 \approx$ the sorts of A_2, then $A_1 {+\cdot} A_2$ is locally-finite.

(29) For all non-empty functions f, g and for every element x of $\prod f$ and for every element y of $\prod g$ holds $x {+\cdot} y \in \prod (f {+\cdot} g)$.

(30) For all non-empty functions f, g and for every element x of $\prod (f {+\cdot} g)$ holds $x \restriction \mathrm{dom}\, g \in \prod g$.

(31) For all non-empty functions f, g such that $f \approx g$ and for every element x of $\prod(f+\cdot g)$ holds $x \restriction \mathrm{dom}\, f \in \prod f$.

(32) Let S_1, S_2 be non empty many sorted signatures, A_1 be a non-empty algebra over S_1, s_1 be an element of $\prod$(the sorts of A_1), A_2 be a non-empty algebra over S_2, and s_2 be an element of $\prod$(the sorts of A_2). If the sorts of $A_1 \approx$ the sorts of A_2, then $s_1+\cdot s_2 \in \prod$(the sorts of $A_1+\cdot A_2$).

(33) Let S_1, S_2 be non empty many sorted signatures, A_1 be a non-empty algebra over S_1, and A_2 be a non-empty algebra over S_2. Suppose the sorts of $A_1 \approx$ the sorts of A_2. Let s be an element of $\prod$(the sorts of $A_1+\cdot A_2$). Then $s\restriction$the carrier of $S_1 \in \prod$(the sorts of A_1) and $s\restriction$the carrier of $S_2 \in \prod$(the sorts of A_2).

(34) Let S_1, S_2 be non void non empty many sorted signatures, A_1 be a non-empty algebra over S_1, and A_2 be a non-empty algebra over S_2. Suppose the sorts of $A_1 \approx$ the sorts of A_2. Let o be an operation symbol of $S_1+\cdot S_2$ and o_2 be an operation symbol of S_2. If $o = o_2$, then $\mathrm{Den}(o, A_1+\cdot A_2) = \mathrm{Den}(o_2, A_2)$.

(35) Let S_1, S_2 be non void non empty many sorted signatures, A_1 be a non-empty algebra over S_1, and A_2 be a non-empty algebra over S_2. Suppose the sorts of $A_1 \approx$ the sorts of A_2 and the characteristics of $A_1 \approx$ the characteristics of A_2. Let o be an operation symbol of $S_1+\cdot S_2$ and o_1 be an operation symbol of S_1. If $o = o_1$, then $\mathrm{Den}(o, A_1+\cdot A_2) = \mathrm{Den}(o_1, A_1)$.

(36) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $S = S_1+\cdot S_2$. Let A_1 be a non-empty circuit of S_1, A_2 be a non-empty circuit of S_2, A be a non-empty circuit of S, s be a state of A, and s_2 be a state of A_2. Suppose $s_2 = s\restriction$the carrier of S_2. Let g be a gate of S and g_2 be a gate of S_2. If $g = g_2$, then $g\,$depends-on-in$\,s = g_2\,$depends-on-in$\,s_2$.

(37) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $S = S_1+\cdot S_2$ and $S_1 \approx S_2$. Let A_1 be a non-empty circuit of S_1, A_2 be a non-empty circuit of S_2, A be a non-empty circuit of S, s be a state of A, and s_1 be a state of A_1. Suppose $s_1 = s\restriction$the carrier of S_1. Let g be a gate of S and g_1 be a gate of S_1. If $g = g_1$, then $g\,$depends-on-in$\,s = g_1\,$depends-on-in$\,s_1$.

(38) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $S = S_1+\cdot S_2$. Let A_1 be a non-empty circuit of S_1, A_2 be a non-empty circuit of S_2, and A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1+\cdot A_2$. Let s be a state of A and v be a vertex of S. Then

(i) for every state s_1 of A_1 such that $s_1 = s\restriction$the carrier of S_1 holds if $v \in \mathrm{InnerVertices}(S_1)$ or $v \in$ the carrier of S_1 and $v \in \mathrm{InputVertices}(S)$, then $(\mathrm{Following}(s))(v) = (\mathrm{Following}(s_1))(v)$, and

(ii) for every state s_2 of A_2 such that $s_2 = s\restriction$the carrier of S_2 holds if $v \in \mathrm{InnerVertices}(S_2)$ or $v \in$ the carrier of S_2 and $v \in \mathrm{InputVertices}(S)$, then $(\mathrm{Following}(s))(v) = (\mathrm{Following}(s_2))(v)$.

(39) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InputVertices}(S_2)$ and $S = S_1+\cdot S_2$. Let A_1 be a non-empty circuit of S_1, A_2 be a non-empty circuit of S_2, and A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1+\cdot A_2$. Let s be a state of A, s_1 be a state of A_1, and s_2 be a state of A_2. Suppose $s_1 = s\restriction$the carrier of S_1 and $s_2 = s\restriction$the carrier of S_2. Then $\mathrm{Following}(s) = \mathrm{Following}(s_1)+\cdot\mathrm{Following}(s_2)$.

(40) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$ and $S = S_1+\cdot S_2$. Let A_1 be a non-empty circuit of S_1, A_2 be a non-empty circuit of S_2, and A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1+\cdot A_2$. Let s be a state of A, s_1 be a state of A_1, and s_2 be a state of A_2. Suppose $s_1 = s\restriction$the carrier of S_1 and $s_2 = s\restriction$the carrier of S_2. Then $\mathrm{Following}(s) = \mathrm{Following}(s_2)+\cdot\mathrm{Following}(s_1)$.

(41) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose InputVertices$(S_1) \subseteq$ InputVertices(S_2) and $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1, A_2 be a non-empty circuit of S_2, and A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let s be a state of A, s_1 be a state of A_1, and s_2 be a state of A_2. Suppose $s_1 = s{\restriction}$the carrier of S_1 and $s_2 = s{\restriction}$the carrier of S_2. Then Following$(s) =$ Following$(s_2){+\cdot}$Following(s_1).

(42) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose InputVertices$(S_2) \subseteq$ InputVertices(S_1) and $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1, A_2 be a non-empty circuit of S_2, and A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let s be a state of A, s_1 be a state of A_1, and s_2 be a state of A_2. Suppose $s_1 = s{\restriction}$the carrier of S_1 and $s_2 = s{\restriction}$the carrier of S_2. Then Following$(s) =$ Following$(s_1){+\cdot}$Following(s_2).

3. Signatures with One Operation

Let A, B be non empty sets and let a be an element of A. Then $B \longmapsto a$ is a function from B into A.

Let f be a set, let p be a finite sequence, and let x be a set. The functor 1GateCircStr(p, f, x) yielding a non void strict many sorted signature is defined by the conditions (Def. 5).

(Def. 5)(i) The carrier of 1GateCircStr$(p, f, x) = \operatorname{rng} p \cup \{x\}$,

(ii) the operation symbols of 1GateCircStr$(p, f, x) = \{\langle p, f\rangle\}$,

(iii) (the arity of 1GateCircStr$(p, f, x))(\langle p, f\rangle) = p$, and

(iv) (the result sort of 1GateCircStr$(p, f, x))(\langle p, f\rangle) = x$.

Let f be a set, let p be a finite sequence, and let x be a set. One can verify that 1GateCircStr(p, f, x) is non empty.

Next we state three propositions:

(43) Let f, x be sets and p be a finite sequence. Then the arity of 1GateCircStr$(p, f, x) = \{\langle p, f\rangle\} \longmapsto p$ and the result sort of 1GateCircStr$(p, f, x) = \{\langle p, f\rangle\} \longmapsto x$.

(44) Let f, x be sets, p be a finite sequence, and g be a gate of 1GateCircStr(p, f, x). Then $g = \langle p, f\rangle$ and Arity$(g) = p$ and the result sort of $g = x$.

(45) For all sets f, x and for every finite sequence p holds InputVertices(1GateCircStr$(p, f, x)) = \operatorname{rng} p \setminus \{x\}$ and InnerVertices(1GateCircStr$(p, f, x)) = \{x\}$.

Let f be a set and let p be a finite sequence. The functor 1GateCircStr(p, f) yields a non void strict many sorted signature and is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of 1GateCircStr$(p, f) = \operatorname{rng} p \cup \{\langle p, f\rangle\}$,

(ii) the operation symbols of 1GateCircStr$(p, f) = \{\langle p, f\rangle\}$,

(iii) (the arity of 1GateCircStr$(p, f))(\langle p, f\rangle) = p$, and

(iv) (the result sort of 1GateCircStr$(p, f))(\langle p, f\rangle) = \langle p, f\rangle$.

Let f be a set and let p be a finite sequence. One can verify that 1GateCircStr(p, f) is non empty.

The following propositions are true:

(46) For every set f and for every finite sequence p holds 1GateCircStr$(p, f) =$ 1GateCircStr$(p, f, \langle p, f\rangle)$.

(47) Let f be a set and p be a finite sequence. Then the arity of 1GateCircStr$(p, f) = \{\langle p, f\rangle\} \longmapsto p$ and the result sort of 1GateCircStr$(p, f) = \{\langle p, f\rangle\} \longmapsto \langle p, f\rangle$.

(48) Let f be a set, p be a finite sequence, and g be a gate of 1GateCircStr(p, f). Then $g = \langle p, f\rangle$ and Arity$(g) = p$ and the result sort of $g = g$.

(49) For every set f and for every finite sequence p holds $\text{InputVertices}(\text{1GateCircStr}(p, f)) = \text{rng}\, p$ and $\text{InnerVertices}(\text{1GateCircStr}(p, f)) = \{\langle p, f\rangle\}$.

(50) For every set f and for every finite sequence p and for every set x such that $x \in \text{rng}\, p$ holds $\text{rk}(x) \in \text{rk}(\langle p, f\rangle)$.

(51) For every set f and for all finite sequences p, q holds $\text{1GateCircStr}(p, f) \approx \text{1GateCircStr}(q, f)$.

4. Unsplit Condition

Let I_1 be a many sorted signature. We say that I_1 is unsplit if and only if:

(Def. 7) The result sort of $I_1 = \text{id}_{\text{the operation symbols of } I_1}$.

We say that I_1 has arity held in gates if and only if:

(Def. 8) For every set g such that $g \in$ the operation symbols of I_1 holds $g = \langle(\text{the arity of } I_1)(g), g_{\mathbf{2}}\rangle$.

We say that I_1 has Boolean denotation held in gates if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let g be a set. Suppose $g \in$ the operation symbols of I_1. Let p be a finite sequence. Suppose $p = (\text{the arity of } I_1)(g)$. Then there exists a function f from $\mathit{Boolean}^{\text{len}\, p}$ into $\mathit{Boolean}$ such that $g = \langle g_{\mathbf{1}}, f\rangle$.

Let S be a non empty many sorted signature and let I_1 be an algebra over S. We say that I_1 has denotation held in gates if and only if:

(Def. 10) For every set g such that $g \in$ the operation symbols of S holds $g = \langle g_{\mathbf{1}}, (\text{the characteristics of } I_1)(g)\rangle$.

Let I_1 be a non empty many sorted signature. We say that I_1 has denotation held in gates if and only if:

(Def. 11) There exists an algebra over I_1 which has denotation held in gates.

Let us observe that every non empty many sorted signature which has Boolean denotation held in gates has also denotation held in gates.

One can prove the following propositions:

(52) Let S be a non empty many sorted signature. Then S is unsplit if and only if for every set o such that $o \in$ the operation symbols of S holds $(\text{the result sort of } S)(o) = o$.

(53) Let S be a non empty many sorted signature. Suppose S is unsplit. Then the operation symbols of $S \subseteq$ the carrier of S.

Let us observe that every non empty many sorted signature which is unsplit is also circuit-like.

The following proposition is true

(54) For every set f and for every finite sequence p holds $\text{1GateCircStr}(p, f)$ is unsplit and has arity held in gates.

Let f be a set and let p be a finite sequence. Note that $\text{1GateCircStr}(p, f)$ is unsplit and has arity held in gates.

Let us observe that there exists a many sorted signature which is unsplit, non void, strict, and non empty and has arity held in gates.

One can prove the following three propositions:

(55) For all unsplit non empty many sorted signatures S_1, S_2 with arity held in gates holds $S_1 \approx S_2$.

(56) Let S_1, S_2 be non empty many sorted signatures, A_1 be an algebra over S_1, and A_2 be an algebra over S_2. Suppose A_1 has denotation held in gates and A_2 has denotation held in gates. Then the characteristics of $A_1 \approx$ the characteristics of A_2.

(57) For all unsplit non empty many sorted signatures S_1, S_2 holds $S_1 +\cdot S_2$ is unsplit.

Let S_1, S_2 be unsplit non empty many sorted signatures. Observe that $S_1 +\cdot S_2$ is unsplit.
Next we state the proposition

(58) For all non empty many sorted signatures S_1, S_2 with arity held in gates holds $S_1 +\cdot S_2$ has arity held in gates.

Let S_1, S_2 be non empty many sorted signatures with arity held in gates. Observe that $S_1 +\cdot S_2$ has arity held in gates.

The following proposition is true

(59) Let S_1, S_2 be non empty many sorted signatures. Suppose S_1 has Boolean denotation held in gates and S_2 has Boolean denotation held in gates. Then $S_1 +\cdot S_2$ has Boolean denotation held in gates.

5. One Gate Circuits

Let n be a natural number. A finite sequence is called a finite sequence with length n if:

(Def. 12) $\operatorname{len} \mathrm{it} = n$.

Let n be a natural number, let X, Y be non empty sets, let f be a function from X^n into Y, let p be a finite sequence with length n, and let x be a set. Let us assume that if $x \in \operatorname{rng} p$, then $X = Y$. The functor $1\mathrm{GateCircuit}(p, f, x)$ yielding a strict non-empty algebra over $1\mathrm{GateCircStr}(p, f, x)$ is defined as follows:

(Def. 13) The sorts of $1\mathrm{GateCircuit}(p, f, x) = (\operatorname{rng} p \longmapsto X) +\cdot (\{x\} \longmapsto Y)$ and (the characteristics of $1\mathrm{GateCircuit}(p, f, x))(\langle p, f \rangle) = f$.

Let n be a natural number, let X be a non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. The functor $1\mathrm{GateCircuit}(p, f)$ yielding a strict non-empty algebra over $1\mathrm{GateCircStr}(p, f)$ is defined as follows:

(Def. 14) The sorts of $1\mathrm{GateCircuit}(p, f) =$ (the carrier of $1\mathrm{GateCircStr}(p, f)) \longmapsto X$ and (the characteristics of $1\mathrm{GateCircuit}(p, f))(\langle p, f \rangle) = f$.

One can prove the following proposition

(60) Let n be a natural number, X be a non empty set, f be a function from X^n into X, and p be a finite sequence with length n. Then $1\mathrm{GateCircuit}(p, f)$ has denotation held in gates and $1\mathrm{GateCircStr}(p, f)$ has denotation held in gates.

Let n be a natural number, let X be a non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. Observe that $1\mathrm{GateCircuit}(p, f)$ has denotation held in gates and $1\mathrm{GateCircStr}(p, f)$ has denotation held in gates.

One can prove the following proposition

(61) Let n be a natural number, p be a finite sequence with length n, and f be a function from $\mathit{Boolean}^n$ into $\mathit{Boolean}$. Then $1\mathrm{GateCircStr}(p, f)$ has Boolean denotation held in gates.

Let n be a natural number, let f be a function from $\mathit{Boolean}^n$ into $\mathit{Boolean}$, and let p be a finite sequence with length n. Observe that $1\mathrm{GateCircStr}(p, f)$ has Boolean denotation held in gates.

Let us note that there exists a many sorted signature which is non empty and has Boolean denotation held in gates.

Let S_1, S_2 be non empty many sorted signatures with Boolean denotation held in gates. Note that $S_1 +\cdot S_2$ has Boolean denotation held in gates.

We now state the proposition

(62) Let n be a natural number, X be a non empty set, f be a function from X^n into X, and p be a finite sequence with length n. Then the characteristics of $1\mathrm{GateCircuit}(p,f) = \{\langle p, f \rangle\} \longmapsto f$ and for every vertex v of $1\mathrm{GateCircStr}(p,f)$ holds (the sorts of $1\mathrm{GateCircuit}(p,f))(v) = X$.

Let n be a natural number, let X be a non empty finite set, let f be a function from X^n into X, and let p be a finite sequence with length n. Observe that $1\mathrm{GateCircuit}(p,f)$ is locally-finite.

Next we state two propositions:

(63) Let n be a natural number, X be a non empty set, f be a function from X^n into X, and p, q be finite sequences with length n. Then $1\mathrm{GateCircuit}(p,f) \approx 1\mathrm{GateCircuit}(q,f)$.

(64) Let n be a natural number, X be a finite non empty set, f be a function from X^n into X, p be a finite sequence with length n, and s be a state of $1\mathrm{GateCircuit}(p,f)$. Then $(\mathrm{Following}(s))(\langle p, f\rangle) = f(s \cdot p)$.

6. Boolean Circuits

Boolean is a finite non empty subset of $\mathbb{N}$.

Let S be a non empty many sorted signature and let I_1 be an algebra over S. We say that I_1 is Boolean if and only if:

(Def. 15) For every vertex v of S holds (the sorts of $I_1)(v) = \mathit{Boolean}$.

One can prove the following proposition

(65) Let S be a non empty many sorted signature and A be an algebra over S. Then A is Boolean if and only if the sorts of $A =$ (the carrier of $S) \longmapsto \mathit{Boolean}$.

Let S be a non empty many sorted signature. Observe that every algebra over S which is Boolean is also non-empty and locally-finite.

The following propositions are true:

(66) Let S be a non empty many sorted signature and A be an algebra over S. Then A is Boolean if and only if rng(the sorts of $A) \subseteq \{\mathit{Boolean}\}$.

(67) Let S_1, S_2 be non empty many sorted signatures, A_1 be an algebra over S_1, and A_2 be an algebra over S_2. Suppose A_1 is Boolean and A_2 is Boolean. Then the sorts of $A_1 \approx$ the sorts of A_2.

(68) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates, A_1 be an algebra over S_1, and A_2 be an algebra over S_2. Suppose A_1 is Boolean and has denotation held in gates and A_2 is Boolean and has denotation held in gates. Then $A_1 \approx A_2$.

Let S be a non empty many sorted signature. Observe that there exists a strict algebra over S which is Boolean.

Next we state three propositions:

(69) Let n be a natural number, f be a function from $\mathit{Boolean}^n$ into $\mathit{Boolean}$, and p be a finite sequence with length n. Then $1\mathrm{GateCircuit}(p,f)$ is Boolean.

(70) Let S_1, S_2 be non empty many sorted signatures, A_1 be a Boolean algebra over S_1, and A_2 be a Boolean algebra over S_2. Then $A_1 {+\cdot} A_2$ is Boolean.

(71) Let S_1, S_2 be non empty many sorted signatures, A_1 be a non-empty algebra over S_1, and A_2 be a non-empty algebra over S_2. Suppose A_1 has denotation held in gates and A_2 has denotation held in gates and the sorts of $A_1 \approx$ the sorts of A_2. Then $A_1 {+\cdot} A_2$ has denotation held in gates.

Let us note that there exists a non empty many sorted signature which is unsplit, non void, and strict and has arity held in gates, denotation held in gates, and Boolean denotation held in gates.

Let S be a non empty many sorted signature with Boolean denotation held in gates. Observe that there exists a strict algebra over S which is Boolean and has denotation held in gates.

Let S_1, S_2 be unsplit non void non empty many sorted signatures with Boolean denotation held in gates, let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates. Note that $A_1 {+\cdot} A_2$ is Boolean and has denotation held in gates.

Let n be a natural number, let X be a finite non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. One can verify that there exists a circuit of $1\mathrm{GateCircStr}(p, f)$ which is strict and non-empty and has denotation held in gates.

Let n be a natural number, let X be a finite non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. Note that $1\mathrm{GateCircuit}(p, f)$ has denotation held in gates.

The following proposition is true

(72) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, A_1 be a Boolean circuit of S_1 with denotation held in gates, A_2 be a Boolean circuit of S_2 with denotation held in gates, s be a state of $A_1 {+\cdot} A_2$, and v be a vertex of $S_1 {+\cdot} S_2$. Then

(i) for every state s_1 of A_1 such that $s_1 = s \restriction$ the carrier of S_1 holds if $v \in \mathrm{InnerVertices}(S_1)$ or $v \in$ the carrier of S_1 and $v \in \mathrm{InputVertices}(S_1 {+\cdot} S_2)$, then $(\mathrm{Following}(s))(v) = (\mathrm{Following}(s_1))(v)$, and

(ii) for every state s_2 of A_2 such that $s_2 = s \restriction$ the carrier of S_2 holds if $v \in \mathrm{InnerVertices}(S_2)$ or $v \in$ the carrier of S_2 and $v \in \mathrm{InputVertices}(S_1 {+\cdot} S_2)$, then $(\mathrm{Following}(s))(v) = (\mathrm{Following}(s_2))(v)$.

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