JOURNAL OF FORMALIZED MATHEMATICS
Volume 11, Released 1999, Published 2003
Inst. of Computer Science, Univ. of Białystok

Property of Complex Functions

Takashi Mitsuishi
Shinshu University
Nagano

Katsumi Wasaki
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. This article introduces properties of complex function, calculations of them, boundedness and constant.

MML Identifier: CFUNCT_1.

WWW: http://mizar.org/JFM/Vol11/cfunct_1.html

The articles [9], [12], [2], [10], [3], [13], [4], [7], [8], [6], [5], [1], and [11] provide the notation and terminology for this paper.

1. Definitions of Complex Functions

For simplicity, we use the following convention: X, Y denote sets, C denotes a non empty set, c denotes an element of C, f, f_1, f_2, f_3, g, g_1 denote partial functions from C to $\mathbb{C}$, and r, q denote elements of $\mathbb{C}$.

A Complex is an element of $\mathbb{C}$.

Let us consider C, f_1, f_2. The functor $\frac{f_1}{f_2}$ yielding a partial function from C to $\mathbb{C}$ is defined by:

(Def. 1) $\mathrm{dom}(\frac{f_1}{f_2}) = \mathrm{dom}\, f_1 \cap (\mathrm{dom}\, f_2 \setminus {f_2}^{-1}(\{0_{\mathbb{C}}\}))$ and for every c such that $c \in \mathrm{dom}(\frac{f_1}{f_2})$ holds $(\frac{f_1}{f_2})_c = (f_1)_c \cdot ((f_2)_c)^{-1}$.

Let us consider C, f. The functor $\frac{1}{f}$ yielding a partial function from C to $\mathbb{C}$ is defined by:

(Def. 2) $\mathrm{dom}(\frac{1}{f}) = \mathrm{dom}\, f \setminus f^{-1}(\{0_{\mathbb{C}}\})$ and for every c such that $c \in \mathrm{dom}(\frac{1}{f})$ holds $(\frac{1}{f})_c = (f_c)^{-1}$.

The following propositions are true:

(3)[1] $\mathrm{dom}(f_1 + f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 + f_2)$ holds $(f_1 + f_2)_c = (f_1)_c + (f_2)_c$.

(4) $\mathrm{dom}(f_1 - f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 - f_2)$ holds $(f_1 - f_2)_c = (f_1)_c - (f_2)_c$.

(5) $\mathrm{dom}(f_1\, f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1\, f_2)$ holds $(f_1\, f_2)_c = (f_1)_c \cdot (f_2)_c$.

(7)[2] $\mathrm{dom}(r\, f) = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}(r\, f)$ holds $(r\, f)_c = r \cdot f_c$.

(9)[3] $\mathrm{dom}(-f) = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}(-f)$ holds $(-f)_c = -f_c$.

(15)[4] $\mathrm{dom}(\frac{1}{g}) \subseteq \mathrm{dom}\, g$ and $\mathrm{dom}\, g \cap (\mathrm{dom}\, g \setminus g^{-1}(\{0_{\mathbb{C}}\})) = \mathrm{dom}\, g \setminus g^{-1}(\{0_{\mathbb{C}}\})$.

[1] The propositions (1) and (2) have been removed.
[2] The proposition (6) has been removed.
[3] The proposition (8) has been removed.
[4] The propositions (10)–(14) have been removed.

© Association of Mizar Users

(16) $\mathrm{dom}(f_1\, f_2) \setminus (f_1\, f_2)^{-1}(\{0_{\mathbb{C}}\}) = (\mathrm{dom} f_1 \setminus {f_1}^{-1}(\{0_{\mathbb{C}}\})) \cap (\mathrm{dom} f_2 \setminus {f_2}^{-1}(\{0_{\mathbb{C}}\}))$.

(17) If $c \in \mathrm{dom}(\frac{1}{f})$, then $f_c \neq 0_{\mathbb{C}}$.

(18) $(\frac{1}{f})^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$.

(19) $|f|^{-1}(\{0\}) = f^{-1}(\{0_{\mathbb{C}}\})$ and $(-f)^{-1}(\{0_{\mathbb{C}}\}) = f^{-1}(\{0_{\mathbb{C}}\})$.

(20) $\mathrm{dom}(\frac{1}{\frac{1}{f}}) = \mathrm{dom}(f \restriction \mathrm{dom}(\frac{1}{f}))$.

(21) If $r \neq 0_{\mathbb{C}}$, then $(r\, f)^{-1}(\{0_{\mathbb{C}}\}) = f^{-1}(\{0_{\mathbb{C}}\})$.

2. Basic Properties of Operations

The following propositions are true:

(22) $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(23) $(f_1\, f_2)\, f_3 = f_1\, (f_2\, f_3)$.

(24) $(f_1 + f_2)\, f_3 = f_1\, f_3 + f_2\, f_3$.

(25) $f_3\, (f_1 + f_2) = f_3\, f_1 + f_3\, f_2$.

(26) $r\, (f_1\, f_2) = (r\, f_1)\, f_2$.

(27) $r\, (f_1\, f_2) = f_1\, (r\, f_2)$.

(28) $(f_1 - f_2)\, f_3 = f_1\, f_3 - f_2\, f_3$.

(29) $f_3\, f_1 - f_3\, f_2 = f_3\, (f_1 - f_2)$.

(30) $r\, (f_1 + f_2) = r\, f_1 + r\, f_2$.

(31) $(r \cdot q)\, f = r\, (q\, f)$.

(32) $r\, (f_1 - f_2) = r\, f_1 - r\, f_2$.

(33) $f_1 - f_2 = (-1_{\mathbb{C}})\, (f_2 - f_1)$.

(34) $f_1 - (f_2 + f_3) = f_1 - f_2 - f_3$.

(35) $1_{\mathbb{C}}\, f = f$.

(36) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(37) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(38) $|f_1\, f_2| = |f_1|\, |f_2|$.

(39) $|r\, f| = |r|\, |f|$.

(40) $-f = (-1_{\mathbb{C}})\, f$.

(41) $--f = f$.

(43)[5] $f_1 - -f_2 = f_1 + f_2$.

(44) $\frac{1}{\frac{1}{f}} = f \restriction \mathrm{dom}(\frac{1}{f})$.

(45) $\frac{1}{f_1\, f_2} = \frac{1}{f_1}\, \frac{1}{f_2}$.

[5] The proposition (42) has been removed.

(46) If $r \neq 0_{\mathbb{C}}$, then $\frac{1}{r\,f} = r^{-1}\,\frac{1}{f}$.

(47) $1_{\mathbb{C}} \neq 0_{\mathbb{C}}$.

(48) $(-1_{\mathbb{C}})^{-1} = -1_{\mathbb{C}}$.

(49) $\frac{1}{-f} = (-1_{\mathbb{C}})\,\frac{1}{f}$.

(50) $\frac{1}{|f|} = |\frac{1}{f}|$.

(51) $\frac{f}{g} = f\,\frac{1}{g}$.

(52) $r\,\frac{g}{f} = \frac{r\,g}{f}$.

(53) $\frac{f}{g}\,g = f{\upharpoonright}\mathrm{dom}(\frac{1}{g})$.

(54) $\frac{f}{g}\,\frac{f_1}{g_1} = \frac{f\,f_1}{g\,g_1}$.

(55) $\frac{1}{\frac{f_1}{f_2}} = \frac{f_2{\upharpoonright}\mathrm{dom}(\frac{1}{f_2})}{f_1}$.

(56) $g\,\frac{f_1}{f_2} = \frac{g\,f_1}{f_2}$.

(57) $\frac{g}{\frac{f_1}{f_2}} = \frac{g\,(f_2{\upharpoonright}\mathrm{dom}(\frac{1}{f_2}))}{f_1}$.

(58) $-\frac{f}{g} = \frac{-f}{g}$ and $\frac{f}{-g} = -\frac{f}{g}$.

(59) $\frac{f_1}{f} + \frac{f_2}{f} = \frac{f_1+f_2}{f}$ and $\frac{f_1}{f} - \frac{f_2}{f} = \frac{f_1-f_2}{f}$.

(60) $\frac{f_1}{f} + \frac{g_1}{g} = \frac{f_1\,g+g_1\,f}{f\,g}$.

(61) $\frac{\frac{f}{g}}{\frac{f_1}{g_1}} = \frac{f\,(g_1{\upharpoonright}\mathrm{dom}(\frac{1}{g_1}))}{g\,f_1}$.

(62) $\frac{f_1}{f} - \frac{g_1}{g} = \frac{f_1\,g-g_1\,f}{f\,g}$.

(63) $|\frac{f_1}{f_2}| = \frac{|f_1|}{|f_2|}$.

(64) $(f_1+f_2){\upharpoonright}X = f_1{\upharpoonright}X + f_2{\upharpoonright}X$ and $(f_1+f_2){\upharpoonright}X = f_1{\upharpoonright}X + f_2$ and $(f_1+f_2){\upharpoonright}X = f_1 + f_2{\upharpoonright}X$.

(65) $(f_1\,f_2){\upharpoonright}X = (f_1{\upharpoonright}X)\,(f_2{\upharpoonright}X)$ and $(f_1\,f_2){\upharpoonright}X = (f_1{\upharpoonright}X)\,f_2$ and $(f_1\,f_2){\upharpoonright}X = f_1\,(f_2{\upharpoonright}X)$.

(66) $(-f){\upharpoonright}X = -f{\upharpoonright}X$ and $\frac{1}{f}{\upharpoonright}X = \frac{1}{f{\upharpoonright}X}$ and $|f|{\upharpoonright}X = |f{\upharpoonright}X|$.

(67) $(f_1-f_2){\upharpoonright}X = f_1{\upharpoonright}X - f_2{\upharpoonright}X$ and $(f_1-f_2){\upharpoonright}X = f_1{\upharpoonright}X - f_2$ and $(f_1-f_2){\upharpoonright}X = f_1 - f_2{\upharpoonright}X$.

(68) $\frac{f_1}{f_2}{\upharpoonright}X = \frac{f_1{\upharpoonright}X}{f_2{\upharpoonright}X}$ and $\frac{f_1}{f_2}{\upharpoonright}X = \frac{f_1{\upharpoonright}X}{f_2}$ and $\frac{f_1}{f_2}{\upharpoonright}X = \frac{f_1}{f_2{\upharpoonright}X}$.

(69) $(r\,f){\upharpoonright}X = r\,(f{\upharpoonright}X)$.

3. Total Partial Functions from a Domain, to Complex

The following propositions are true:

(70)(i) f_1 is total and f_2 is total iff $f_1 + f_2$ is total,

(ii) f_1 is total and f_2 is total iff $f_1 - f_2$ is total, and

(iii) f_1 is total and f_2 is total iff $f_1\, f_2$ is total.

(71) f is total iff $r\, f$ is total.

(72) f is total iff $-f$ is total.

(73) f is total iff $|f|$ is total.

(74) $\frac{1}{f}$ is total iff $f^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$ and f is total.

(75) f_1 is total and ${f_2}^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$ and f_2 is total iff $\frac{f_1}{f_2}$ is total.

(76) If f_1 is total and f_2 is total, then $(f_1 + f_2)_c = (f_1)_c + (f_2)_c$ and $(f_1 - f_2)_c = (f_1)_c - (f_2)_c$ and $(f_1\, f_2)_c = (f_1)_c \cdot (f_2)_c$.

(77) If f is total, then $(r\, f)_c = r \cdot f_c$.

(78) If f is total, then $(-f)_c = -f_c$ and $|f|(c) = |f_c|$.

(79) If $\frac{1}{f}$ is total, then $(\frac{1}{f})_c = (f_c)^{-1}$.

(80) If f_1 is total and $\frac{1}{f_2}$ is total, then $(\frac{f_1}{f_2})_c = (f_1)_c \cdot ((f_2)_c)^{-1}$.

4. Bounded and Constant Partial Functions from a Domain, to Complex

Let us consider C, f, Y. We say that f is bounded on Y if and only if:

(Def. 3) $|f|$ is bounded on Y.

We now state a number of propositions:

(81) f is bounded on Y iff there exists a real number p such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $|f_c| \leq p$.

(82) If $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(83) If X misses $\operatorname{dom} f$, then f is bounded on X.

(84) If f is bounded on Y, then $r\, f$ is bounded on Y.

(85) $|f|$ is lower bounded on X.

(86) If f is bounded on Y, then $|f|$ is bounded on Y and $-f$ is bounded on Y.

(87) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(88) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1\, f_2$ is bounded on $X \cap Y$ and $f_1 - f_2$ is bounded on $X \cap Y$.

(89) If f is bounded on X and bounded on Y, then f is bounded on $X \cup Y$.

(90) Suppose f_1 is a constant on X and f_2 is a constant on Y. Then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$ and $f_1\, f_2$ is a constant on $X \cap Y$.

(91) If f is a constant on Y, then $q\, f$ is a constant on Y.

(92) If f is a constant on Y, then $|f|$ is a constant on Y and $-f$ is a constant on Y.

(93) If f is a constant on Y, then f is bounded on Y.

(94) If f is a constant on Y, then for every r holds $r f$ is bounded on Y and $-f$ is bounded on Y and $|f|$ is bounded on Y.

(95) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(96) Suppose f_1 is bounded on X and f_2 is a constant on Y. Then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$ and f_1 f_2 is bounded on $X \cap Y$.

REFERENCES

[1] Agnieszka Banachowicz and Anna Winnicka. Complex sequences. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/comseq_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[5] Czesław Byliński. The complex numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/complex1.html.

[6] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[7] Jarosław Kotowicz. Partial functions from a domain to a domain. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/partfun2.html.

[8] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rfunct_1.html.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[10] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[11] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[13] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received December 7, 1999

Published January 2, 2004