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Property of Complex Sequence and Continuity of Complex Function

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Summary. This article introduces properties of complex sequence and continuity of complex function. The first section shows convergence of complex sequence and constant complex sequence. In the next section, definition of continuity of complex function and properties of continuous complex function are shown.

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The articles [11], [14], [15], [3], [12], [2], [5], [4], [16], [7], [6], [13], [8], [1], [9], and [10] provide the notation and terminology for this paper.

1. Complex Sequence

For simplicity, we follow the rules: n, m, k are natural numbers, x, X, X_1 are sets, g, x_0, x_1, x_2 are elements of $\mathbb{C}$, s_1, s_2, s_3, s_4, s_5, s_6 are complex sequences, Y is a subset of $\mathbb{C}$, f, f_1, f_2, h, h_1, h_2 are partial functions from $\mathbb{C}$ to $\mathbb{C}$, r, s are real numbers, and N_1 is an increasing sequence of naturals.

Let us consider h, s_3. Let us assume that $\operatorname{rng} s_3 \subseteq \operatorname{dom} h$. The functor $h \cdot s_3$ yields a complex sequence and is defined as follows:

(Def. 1) $h \cdot s_3 = (h$ **qua** function$) \cdot (s_3)$.

Let us consider f, x_0. We say that f is continuous in x_0 if and only if:

(Def. 2) $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

We now state a number of propositions:

(2)[1] $s_4 = s_5 - s_6$ iff for every n holds $s_4(n) = s_5(n) - s_6(n)$.

(3) $\operatorname{rng}(s_3 \uparrow n) \subseteq \operatorname{rng} s_3$.

(4) If $\operatorname{rng} s_3 \subseteq \operatorname{dom} f$, then $s_3(n) \in \operatorname{dom} f$.

(5) $x \in \operatorname{rng} s_3$ iff there exists n such that $x = s_3(n)$.

(6) $s_3(n) \in \operatorname{rng} s_3$.

(7) If s_4 is a subsequence of s_3, then $\operatorname{rng} s_4 \subseteq \operatorname{rng} s_3$.

[1] The proposition (1) has been removed.

(8) If s_4 is a subsequence of s_3 and s_3 is non-zero, then s_4 is non-zero.

(9) $(s_4+s_5)\,N_1 = s_4\,N_1+s_5\,N_1$ and $(s_4-s_5)\,N_1 = s_4\,N_1-s_5\,N_1$ and $(s_4\,s_5)\,N_1 = s_4\,N_1\,(s_5\,N_1)$.

(10) $(g\,s_3)\,N_1 = g\,(s_3\,N_1)$.

(11) $(-s_3)\,N_1 = -s_3\,N_1$ and $|s_3|\cdot N_1 = |s_3\,N_1|$.

(12) $(s_3\,N_1)^{-1} = {s_3}^{-1}\,N_1$.

(13) $(s_4/s_3)\,N_1 = (s_4\,N_1)/(s_3\,N_1)$.

(14) If for every n holds $s_3(n) \in Y$, then $\operatorname{rng} s_3 \subseteq Y$.

(16)[2] If $\operatorname{rng} s_3 \subseteq \operatorname{dom} f$, then $(f\cdot s_3)(n) = f_{s_3(n)}$.

(17) If $\operatorname{rng} s_3 \subseteq \operatorname{dom} f$, then $(f\cdot s_3)\uparrow n = f\cdot(s_3\uparrow n)$.

(18) If $\operatorname{rng} s_3 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1+h_2)\cdot s_3 = h_1\cdot s_3 + h_2\cdot s_3$ and $(h_1-h_2)\cdot s_3 = h_1\cdot s_3 - h_2\cdot s_3$ and $(h_1\,h_2)\cdot s_3 = (h_1\cdot s_3)\,(h_2\cdot s_3)$.

(19) If $\operatorname{rng} s_3 \subseteq \operatorname{dom} h$, then $(g\,h)\cdot s_3 = g\,(h\cdot s_3)$.

(20) If $\operatorname{rng} s_3 \subseteq \operatorname{dom} h$, then $-h\cdot s_3 = (-h)\cdot s_3$.

(21) If $\operatorname{rng} s_3 \subseteq \operatorname{dom}(\frac{1}{h})$, then $h\cdot s_3$ is non-zero.

(22) If $\operatorname{rng} s_3 \subseteq \operatorname{dom}(\frac{1}{h})$, then $\frac{1}{h}\cdot s_3 = (h\cdot s_3)^{-1}$.

(23) If $\operatorname{rng} s_3 \subseteq \operatorname{dom} h$, then $\Re((h\cdot s_3)\,N_1) = \Re(h\cdot(s_3\,N_1))$.

(24) If $\operatorname{rng} s_3 \subseteq \operatorname{dom} h$, then $\Im((h\cdot s_3)\,N_1) = \Im(h\cdot(s_3\,N_1))$.

(25) If $\operatorname{rng} s_3 \subseteq \operatorname{dom} h$, then $(h\cdot s_3)\,N_1 = h\cdot(s_3\,N_1)$.

(26) If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$ and s_5 is a subsequence of s_4, then $h\cdot s_5$ is a subsequence of $h\cdot s_4$.

(27) If h is total, then $(h\cdot s_3)(n) = h_{s_3(n)}$.

(28) If h is total, then $h\cdot(s_3\uparrow n) = (h\cdot s_3)\uparrow n$.

(29) If h_1 is total and h_2 is total, then $(h_1+h_2)\cdot s_3 = h_1\cdot s_3 + h_2\cdot s_3$ and $(h_1-h_2)\cdot s_3 = h_1\cdot s_3 - h_2\cdot s_3$ and $(h_1\,h_2)\cdot s_3 = (h_1\cdot s_3)\,(h_2\cdot s_3)$.

(30) If h is total, then $(g\,h)\cdot s_3 = g\,(h\cdot s_3)$.

(31) If $\operatorname{rng} s_3 \subseteq \operatorname{dom}(h{\upharpoonright}X)$, then $(h{\upharpoonright}X)\cdot s_3 = h\cdot s_3$.

(32) If $\operatorname{rng} s_3 \subseteq \operatorname{dom}(h{\upharpoonright}X)$ and if $\operatorname{rng} s_3 \subseteq \operatorname{dom}(h{\upharpoonright}Y)$ or $X \subseteq Y$, then $(h{\upharpoonright}X)\cdot s_3 = (h{\upharpoonright}Y)\cdot s_3$.

(33) If $\operatorname{rng} s_3 \subseteq \operatorname{dom}(h{\upharpoonright}X)$ and $h^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$, then $(\frac{1}{h}{\upharpoonright}X)\cdot s_3 = ((h{\upharpoonright}X)\cdot s_3)^{-1}$.

Let us consider s_3. Let us observe that s_3 is constant if and only if:

(Def. 4)[3] There exists g such that for every n holds $s_3(n) = g$.

One can prove the following propositions:

(34) s_3 is constant iff there exists g such that $\operatorname{rng} s_3 = \{g\}$.

(35) s_3 is constant iff for every n holds $s_3(n) = s_3(n+1)$.

(36) s_3 is constant iff for all n, k holds $s_3(n) = s_3(n+k)$.

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[2] The proposition (15) has been removed.

[3] The definition (Def. 3) has been removed.

(37) s_3 is constant iff for all n, m holds $s_3(n) = s_3(m)$.

(38) $s_3 \uparrow k$ is a subsequence of s_3.

(39) If s_4 is a subsequence of s_3 and s_3 is convergent, then s_4 is convergent.

(40) If s_4 is a subsequence of s_3 and s_3 is convergent, then $\lim s_4 = \lim s_3$.

(41) If s_3 is convergent and there exists k such that for every n such that $k \leq n$ holds $s_4(n) = s_3(n)$, then s_4 is convergent.

(42) If s_3 is convergent and there exists k such that for every n such that $k \leq n$ holds $s_4(n) = s_3(n)$, then $\lim s_3 = \lim s_4$.

(43) If s_3 is convergent, then $s_3 \uparrow k$ is convergent and $\lim(s_3 \uparrow k) = \lim s_3$.

(44) If s_3 is convergent and there exists k such that $s_3 = s_4 \uparrow k$, then s_4 is convergent.

(45) If s_3 is convergent and there exists k such that $s_3 = s_4 \uparrow k$, then $\lim s_4 = \lim s_3$.

(46) If s_3 is convergent and $\lim s_3 \neq 0_{\mathbb{C}}$, then there exists k such that $s_3 \uparrow k$ is non-zero.

(47) If s_3 is convergent and $\lim s_3 \neq 0_{\mathbb{C}}$, then there exists s_4 which is a subsequence of s_3 and non-zero.

(48) If s_3 is constant, then s_3 is convergent.

(49) If s_3 is constant and $g \in \operatorname{rng} s_3$ or s_3 is constant and there exists n such that $s_3(n) = g$, then $\lim s_3 = g$.

(50) If s_3 is constant, then for every n holds $\lim s_3 = s_3(n)$.

(51) If s_3 is convergent and $\lim s_3 \neq 0_{\mathbb{C}}$, then for every s_4 such that s_4 is a subsequence of s_3 and non-zero holds $\lim({s_4}^{-1}) = (\lim s_3)^{-1}$.

(52) If s_3 is constant and s_4 is convergent, then $\lim(s_3 + s_4) = s_3(0) + \lim s_4$ and $\lim(s_3 - s_4) = s_3(0) - \lim s_4$ and $\lim(s_4 - s_3) = \lim s_4 - s_3(0)$ and $\lim(s_3\, s_4) = s_3(0) \cdot \lim s_4$.

The scheme *CompSeqChoice* concerns a binary predicate $\mathcal{P}$, and states that:

There exists s_1 such that for every n holds $\mathcal{P}[n, s_1(n)]$

provided the parameters meet the following condition:

- For every n there exists g such that $\mathcal{P}[n, g]$.

2. Continuity of Complex Sequence

The following propositions are true:

(53) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ and for every n holds $s_1(n) \neq x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

(54) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(55) Suppose f_1 is continuous in x_0 and f_2 is continuous in x_0. Then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0 and $f_1\, f_2$ is continuous in x_0.

(56) If f is continuous in x_0, then $g\, f$ is continuous in x_0.

(57) If f is continuous in x_0, then $-f$ is continuous in x_0.

(58) If f is continuous in x_0 and $f_{x_0} \neq 0_{\mathbb{C}}$, then $\frac{1}{f}$ is continuous in x_0.

(59) If f_1 is continuous in x_0 and $(f_1)_{x_0} \neq 0_{\mathbb{C}}$ and f_2 is continuous in x_0, then $\frac{f_2}{f_1}$ is continuous in x_0.

Let us consider f, X. We say that f is continuous on X if and only if:

(Def. 5) $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

The following propositions are true:

(60) Let given X, f. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds $f \cdot s_1$ is convergent and $f_{\lim s_1} = \lim(f \cdot s_1)$.

(61) f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(62) f is continuous on X iff $f \restriction X$ is continuous on X.

(63) If f is continuous on X and $X_1 \subseteq X$, then f is continuous on X_1.

(64) If $x_0 \in \operatorname{dom} f$, then f is continuous on $\{x_0\}$.

(65) Let given X, f_1, f_2. Suppose f_1 is continuous on X and f_2 is continuous on X. Then $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X and $f_1\, f_2$ is continuous on X.

(66) Let given X, X_1, f_1, f_2. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$ and $f_1\, f_2$ is continuous on $X \cap X_1$.

(67) For all g, X, f such that f is continuous on X holds $g\, f$ is continuous on X.

(68) If f is continuous on X, then $-f$ is continuous on X.

(69) If f is continuous on X and $f^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$, then $\frac{1}{f}$ is continuous on X.

(70) If f is continuous on X and $(f \restriction X)^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$, then $\frac{1}{f}$ is continuous on X.

(71) If f_1 is continuous on X and $f_1{}^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$ and f_2 is continuous on X, then $\frac{f_2}{f_1}$ is continuous on X.

(72) If f is total and for all x_1, x_2 holds $f_{x_1 + x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0, then f is continuous on $\mathbb{C}$.

Let us consider X. We say that X is compact if and only if:

(Def. 6) For every s_1 such that $\operatorname{rng} s_1 \subseteq X$ there exists s_2 such that s_2 is a subsequence of s_1 and convergent and $\lim s_2 \in X$.

We now state two propositions:

(73) For every f such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(74) If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y, then $f^{\circ} Y$ is compact.

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