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Cartesian Categories

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Summary. We define and prove some simple facts on Cartesian categories and its duals co-Cartesian categories. The Cartesian category is defined as a category with the fixed terminal object, the fixed projections, and the binary products. Category C has finite products if and only if C has a terminal object and for every pair a, b of objects of C the product $a \times b$ exists. We say that a category C has a finite product if every finite family of objects of C has a product. Our work is based on ideas of [10], where the algebraic properties of the proof theory are investigated. The terminal object of a Cartesian category C is denoted by $\mathbf{1}_C$. The binary product of a and b is written as $a \times b$. The projections of the product are written as $pr_1(a, b)$ and as $pr_2(a, b)$. We define the products $f \times g$ of arrows $f : a \to a'$ and $g : b \to b'$ as $< f \cdot pr_1, g \cdot pr_2 >: a \times b \to a' \times b'$.

Co-Cartesian category is defined dually to the Cartesian category. Dual to a terminal object is an initial object, and to products are coproducts. The initial object of a Cartesian category C is written as $\mathbf{0}_C$. Binary coproduct of a and b is written as $a + b$. Injections of the coproduct are written as $in_1(a, b)$ and as $in_2(a, b)$.

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The articles [12], [5], [13], [8], [11], [14], [2], [3], [9], [1], [6], [4], and [7] provide the notation and terminology for this paper.

1. Preliminaries

In this paper o, m, r denote sets.

Let us consider o, m, r.

(Def. 1) $[\langle o, m\rangle \mapsto r]$ is a function from $[: \{o\}, \{m\} :]$ into $\{r\}$.

Let C be a category and let a, b be objects of C. Let us observe that a and b are isomorphic if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $\hom(a, b) \neq \emptyset$,

(ii) $\hom(b, a) \neq \emptyset$, and

(iii) there exists a morphism f from a to b and there exists a morphism f' from b to a such that $f \cdot f' = \mathrm{id}_b$ and $f' \cdot f = \mathrm{id}_a$.

2. Cartesian Categories

Let C be a category. We say that C has finite product if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let I be a finite set and F be a function from I into the objects of C. Then there exists an object a of C and there exists a projections family F' from a onto I such that $\mathrm{cod}_\kappa F'(\kappa) = F$ and a is a product w.r.t. F'.

The following proposition is true

(1) Let C be a category. Then C has finite product if and only if the following conditions are satisfied:

(i) there exists an object of C which is terminal, and

(ii) for all objects a, b of C there exists an object c of C and there exist morphisms p_1, p_2 of C such that $\mathrm{dom}\, p_1 = c$ and $\mathrm{dom}\, p_2 = c$ and $\mathrm{cod}\, p_1 = a$ and $\mathrm{cod}\, p_2 = b$ and c is a product w.r.t. p_1 and p_2.

We consider Cartesian category structures as extensions of category structure as systems
⟨ objects, morphisms, a dom-map, a cod-map, a composition, an id-map, a terminal, a product, a 1st-projection, a 2nd-projection ⟩,
where the objects and the morphisms constitute non empty sets, the dom-map and the cod-map are functions from the morphisms into the objects, the composition is a partial function from [: the morphisms, the morphisms :] to the morphisms, the id-map is a function from the objects into the morphisms, the terminal is an element of the objects, the product is a function from [: the objects, the objects :] into the objects, and the 1st-projection and the 2nd-projection are functions from [: the objects, the objects :] into the morphisms.

Let C be a Cartesian category structure. The functor $\mathbf{1}_C$ yields an object of C and is defined by:

(Def. 4) $\mathbf{1}_C =$ the terminal of C.

Let a, b be objects of C. The functor $a \times b$ yields an object of C and is defined by:

(Def. 5) $a \times b =$ (the product of C)($\langle a, b \rangle$).

The functor $\pi_1(a \times b)$ yields a morphism of C and is defined by:

(Def. 6) $\pi_1(a \times b) =$ (the 1st-projection of C)($\langle a, b \rangle$).

The functor $\pi_2(a \times b)$ yielding a morphism of C is defined as follows:

(Def. 7) $\pi_2(a \times b) =$ (the 2nd-projection of C)($\langle a, b \rangle$).

Let us consider o, m. The functor $\dot{\circlearrowleft}_{\mathrm{c}}(o,m)$ yielding a strict Cartesian category structure is defined as follows:

(Def. 8) $\dot{\circlearrowleft}_{\mathrm{c}}(o,m) = \langle \{o\}, \{m\}, \{m\} \longmapsto o, \{m\} \longmapsto o, \langle m,m \rangle \longmapsto m, \{o\} \longmapsto m, \mathrm{Extract}(o), [\langle o,o \rangle \mapsto o], [\langle o,o \rangle \mapsto m], \{o\} 0 \rangle$.

Next we state the proposition

(2) The category structure of $\dot{\circlearrowleft}_{\mathrm{c}}(o,m) = \dot{\circlearrowleft}(o,m)$.

Let us observe that there exists a Cartesian category structure which is strict and category-like.
Let o, m be sets. Observe that $\dot{\circlearrowleft}_{\mathrm{c}}(o,m)$ is category-like.
One can prove the following propositions:

(3) For every object a of $\dot{\circlearrowleft}_{\mathrm{c}}(o,m)$ holds $a = o$.

(4) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o,m)$ holds $a = b$.

(5) For every morphism f of $\dot{\circlearrowleft}_{\mathrm{c}}(o,m)$ holds $f = m$.

(6) For all morphisms f, g of $\dot{\circlearrowleft}_{\mathrm{c}}(o,m)$ holds $f = g$.

(7) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o,m)$ and for every morphism f of $\dot{\circlearrowleft}_{\mathrm{c}}(o,m)$ holds $f \in \hom(a,b)$.

(8) For all objects a, b of $\dot{\circlearrowright}_{\rm c}(o,m)$ holds every morphism of $\dot{\circlearrowright}_{\rm c}(o,m)$ is a morphism from a to b.

(9) For all objects a, b of $\dot{\circlearrowright}_{\rm c}(o,m)$ holds $\hom(a,b) \neq \emptyset$.

(10) Every object of $\dot{\circlearrowright}_{\rm c}(o,m)$ is terminal.

(11) For every object c of $\dot{\circlearrowright}_{\rm c}(o,m)$ and for all morphisms p_1, p_2 of $\dot{\circlearrowright}_{\rm c}(o,m)$ holds c is a product w.r.t. p_1 and p_2.

Let I_1 be a category-like Cartesian category structure. We say that I_1 is Cartesian if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) The terminal of I_1 is terminal, and

(ii) for all objects a, b of I_1 holds cod(the 1st-projection of I_1)$(\langle a, b\rangle) = a$ and cod(the 2nd-projection of I_1)$(\langle a, b\rangle) = b$ and (the product of I_1)$(\langle a, b\rangle)$ is a product w.r.t. (the 1st-projection of I_1)$(\langle a, b\rangle)$ and (the 2nd-projection of I_1)$(\langle a, b\rangle)$.

One can prove the following proposition

(12) For all sets o, m holds $\dot{\circlearrowright}_{\rm c}(o,m)$ is Cartesian.

One can check that there exists a category-like Cartesian category structure which is strict and Cartesian.

A Cartesian category is a Cartesian category-like Cartesian category structure.

We follow the rules: C denotes a Cartesian category and a, b, c, d, e, s denote objects of C.

Next we state three propositions:

(13) $\mathbf{1}_C$ is terminal.

(14) For all morphisms f_1, f_2 from a to $\mathbf{1}_C$ holds $f_1 = f_2$.

(15) $\hom(a,\mathbf{1}_C) \neq \emptyset$.

Let us consider C, a.

(Def. 10) $\mathrm{term}\,a$ is a morphism from a to $\mathbf{1}_C$.

We now state several propositions:

(16) $\mathrm{term}\,a = |^a(\mathbf{1}_C)$.

(17) $\mathrm{dom}\,\mathrm{term}\,a = a$ and $\mathrm{cod}\,\mathrm{term}\,a = \mathbf{1}_C$.

(18) $\hom(a,\mathbf{1}_C) = \{\mathrm{term}\,a\}$.

(19) $\mathrm{dom}\,\pi_1(a \times b) = a \times b$ and $\mathrm{cod}\,\pi_1(a \times b) = a$.

(20) $\mathrm{dom}\,\pi_2(a \times b) = a \times b$ and $\mathrm{cod}\,\pi_2(a \times b) = b$.

Let us consider C, a, b. Then $\pi_1(a \times b)$ is a morphism from $a \times b$ to a. Then $\pi_2(a \times b)$ is a morphism from $a \times b$ to b.

The following four propositions are true:

(21) $\hom(a \times b, a) \neq \emptyset$ and $\hom(a \times b, b) \neq \emptyset$.

(22) $a \times b$ is a product w.r.t. $\pi_1(a \times b)$ and $\pi_2(a \times b)$.

(23) C has finite product.

(24) If $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$, then $\pi_1(a \times b)$ is retraction and $\pi_2(a \times b)$ is retraction.

Let us consider C, a, b, c, let f be a morphism from c to a, and let g be a morphism from c to b. Let us assume that $\hom(c,a) \neq \emptyset$ and $\hom(c,b) \neq \emptyset$. The functor $\langle f,g\rangle$ yielding a morphism from c to $a \times b$ is defined as follows:

(Def. 11) $\pi_1(a \times b) \cdot \langle f,g\rangle = f$ and $\pi_2(a \times b) \cdot \langle f,g\rangle = g$.

We now state several propositions:

(25) If $\hom(c,a) \neq \emptyset$ and $\hom(c,b) \neq \emptyset$, then $\hom(c,a \times b) \neq \emptyset$.

(26) $\langle \pi_1(a \times b), \pi_2(a \times b)\rangle = \mathrm{id}_{a \times b}$.

(27) Let f be a morphism from c to a, g be a morphism from c to b, and h be a morphism from d to c. If $\hom(c,a) \neq \emptyset$ and $\hom(c,b) \neq \emptyset$ and $\hom(d,c) \neq \emptyset$, then $\langle f \cdot h, g \cdot h\rangle = \langle f,g\rangle \cdot h$.

(28) Let f, k be morphisms from c to a and g, h be morphisms from c to b. If $\hom(c,a) \neq \emptyset$ and $\hom(c,b) \neq \emptyset$ and $\langle f,g\rangle = \langle k,h\rangle$, then $f = k$ and $g = h$.

(29) Let f be a morphism from c to a and g be a morphism from c to b. If $\hom(c,a) \neq \emptyset$ and if $\hom(c,b) \neq \emptyset$ and if f is monic or g is monic, then $\langle f,g\rangle$ is monic.

(30) $\hom(a, a \times \mathbf{1}_C) \neq \emptyset$ and $\hom(a, \mathbf{1}_C \times a) \neq \emptyset$.

Let us consider C, a. The functor $\lambda(a)$ yielding a morphism from $\mathbf{1}_C \times a$ to a is defined as follows:

(Def. 12) $\lambda(a) = \pi_2(\mathbf{1}_C \times a)$.

The functor $\lambda^{-1}(a)$ yields a morphism from a to $\mathbf{1}_C \times a$ and is defined by:

(Def. 13) $\lambda^{-1}(a) = \langle \mathrm{term}\, a, \mathrm{id}_a\rangle$.

The functor $\rho(a)$ yielding a morphism from $a \times \mathbf{1}_C$ to a is defined as follows:

(Def. 14) $\rho(a) = \pi_1(a \times \mathbf{1}_C)$.

The functor $\rho^{-1}(a)$ yields a morphism from a to $a \times \mathbf{1}_C$ and is defined by:

(Def. 15) $\rho^{-1}(a) = \langle \mathrm{id}_a, \mathrm{term}\, a\rangle$.

Next we state two propositions:

(31) $\lambda(a) \cdot \lambda^{-1}(a) = \mathrm{id}_a$ and $\lambda^{-1}(a) \cdot \lambda(a) = \mathrm{id}_{\mathbf{1}_C \times a}$ and $\rho(a) \cdot \rho^{-1}(a) = \mathrm{id}_a$ and $\rho^{-1}(a) \cdot \rho(a) = \mathrm{id}_{a \times \mathbf{1}_C}$.

(32) a and $a \times \mathbf{1}_C$ are isomorphic and a and $\mathbf{1}_C \times a$ are isomorphic.

Let us consider C, a, b. The functor $\mathrm{Switch}(a)$ yields a morphism from $a \times b$ to $b \times a$ and is defined by:

(Def. 16) $\mathrm{Switch}(a) = \langle \pi_2(a \times b), \pi_1(a \times b)\rangle$.

We now state three propositions:

(33) $\hom(a \times b, b \times a) \neq \emptyset$.

(34) $\mathrm{Switch}(a) \cdot \mathrm{Switch}(b) = \mathrm{id}_{b \times a}$.

(35) $a \times b$ and $b \times a$ are isomorphic.

Let us consider C, a. The functor $\Delta(a)$ yields a morphism from a to $a \times a$ and is defined as follows:

(Def. 17) $\Delta(a) = \langle \mathrm{id}_a, \mathrm{id}_a\rangle$.

We now state two propositions:

(36) $\hom(a, a \times a) \neq \emptyset$.

(37) For every morphism f from a to b such that $\hom(a,b) \neq \emptyset$ holds $\langle f, f \rangle = \Delta(b) \cdot f$.

Let us consider C, a, b, c. The functor $\alpha((a,b),c)$ yielding a morphism from $a \times b \times c$ to $a \times (b \times c)$ is defined as follows:

(Def. 18) $\alpha((a,b),c) = \langle \pi_1(a \times b) \cdot \pi_1((a \times b) \times c), \langle \pi_2(a \times b) \cdot \pi_1((a \times b) \times c), \pi_2((a \times b) \times c) \rangle \rangle$.

The functor $\alpha(a,(b,c))$ yields a morphism from $a \times (b \times c)$ to $a \times b \times c$ and is defined by:

(Def. 19) $\alpha(a,(b,c)) = \langle \langle \pi_1(a \times (b \times c)), \pi_1(b \times c) \cdot \pi_2(a \times (b \times c)) \rangle, \pi_2(b \times c) \cdot \pi_2(a \times (b \times c)) \rangle$.

We now state three propositions:

(38) $\hom(a \times b \times c, a \times (b \times c)) \neq \emptyset$ and $\hom(a \times (b \times c), a \times b \times c) \neq \emptyset$.

(39) $\alpha((a,b),c) \cdot \alpha(a,(b,c)) = \mathrm{id}_{a \times (b \times c)}$ and $\alpha(a,(b,c)) \cdot \alpha((a,b),c) = \mathrm{id}_{a \times b \times c}$.

(40) $(a \times b) \times c$ and $a \times (b \times c)$ are isomorphic.

Let us consider C, a, b, c, d, let f be a morphism from a to b, and let g be a morphism from c to d. The functor $f \times g$ yielding a morphism from $a \times c$ to $b \times d$ is defined by:

(Def. 20) $f \times g = \langle f \cdot \pi_1(a \times c), g \cdot \pi_2(a \times c) \rangle$.

The following propositions are true:

(41) If $\hom(a,c) \neq \emptyset$ and $\hom(b,d) \neq \emptyset$, then $\hom(a \times b, c \times d) \neq \emptyset$.

(42) $\mathrm{id}_a \times \mathrm{id}_b = \mathrm{id}_{a \times b}$.

(43) Let f be a morphism from a to b, h be a morphism from c to d, g be a morphism from e to a, and k be a morphism from e to c. If $\hom(a,b) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$ and $\hom(e,a) \neq \emptyset$ and $\hom(e,c) \neq \emptyset$, then $(f \times h) \cdot \langle g, k \rangle = \langle f \cdot g, h \cdot k \rangle$.

(44) Let f be a morphism from c to a and g be a morphism from c to b. If $\hom(c,a) \neq \emptyset$ and $\hom(c,b) \neq \emptyset$, then $\langle f, g \rangle = (f \times g) \cdot \Delta(c)$.

(45) Let f be a morphism from a to b, h be a morphism from c to d, g be a morphism from e to a, and k be a morphism from s to c. If $\hom(a,b) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$ and $\hom(e,a) \neq \emptyset$ and $\hom(s,c) \neq \emptyset$, then $(f \times h) \cdot (g \times k) = (f \cdot g) \times (h \cdot k)$.

3. Co-Cartesian Categories

Let C be a category. We say that C has finite coproduct if and only if the condition (Def. 21) is satisfied.

(Def. 21) Let I be a finite set and F be a function from I into the objects of C. Then there exists an object a of C and there exists an injections family F' into a on I such that $\mathrm{dom}_\kappa F'(\kappa) = F$ and a is a coproduct w.r.t. F'.

The following proposition is true

(46) Let C be a category. Then C has finite coproduct if and only if the following conditions are satisfied:

(i) there exists an object of C which is initial, and

(ii) for all objects a, b of C there exists an object c of C and there exist morphisms i_1, i_2 of C such that $\mathrm{dom}\, i_1 = a$ and $\mathrm{dom}\, i_2 = b$ and $\mathrm{cod}\, i_1 = c$ and $\mathrm{cod}\, i_2 = c$ and c is a coproduct w.r.t. i_1 and i_2.

We introduce cocartesian category structures which are extensions of category structure and are systems

⟨ objects, morphisms, a dom-map, a cod-map, a composition, an id-map, a initial, a coproduct, a 1st-coprojection, a 2nd-coprojection ⟩,

where the objects and the morphisms constitute non empty sets, the dom-map and the cod-map are functions from the morphisms into the objects, the composition is a partial function from [: the morphisms, the morphisms :] to the morphisms, the id-map is a function from the objects into the morphisms, the initial is an element of the objects, the coproduct is a function from [: the objects, the objects :] into the objects, and the 1st-coprojection and the 2nd-coprojection are functions from [: the objects, the objects :] into the morphisms.

Let C be a cocartesian category structure. The functor $\mathbf{0}_C$ yields an object of C and is defined as follows:

(Def. 22) $\mathbf{0}_C =$ the initial of C.

Let a, b be objects of C. The functor $a+b$ yields an object of C and is defined by:

(Def. 23) $a+b =$ (the coproduct of C)($\langle a, b\rangle$).

The functor $\mathrm{in}_1(a+b)$ yields a morphism of C and is defined as follows:

(Def. 24) $\mathrm{in}_1(a+b) =$ (the 1st-coprojection of C)($\langle a, b\rangle$).

The functor $\mathrm{in}_2(a+b)$ yields a morphism of C and is defined by:

(Def. 25) $\mathrm{in}_2(a+b) =$ (the 2nd-coprojection of C)($\langle a, b\rangle$).

Let us consider o, m. The functor $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ yields a strict cocartesian category structure and is defined as follows:

(Def. 26) $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m) = \langle \{o\}, \{m\}, \{m\} \longmapsto o, \{m\} \longmapsto o, \langle m,m\rangle \longmapsto m, \{o\} \longmapsto m, \mathrm{Extract}(o), [\langle o,o\rangle \mapsto o], [\langle o,o\rangle \mapsto m], \{o\}0\rangle$.

We now state the proposition

(47) The category structure of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m) = \circlearrowleft(o,m)$.

Let us note that there exists a cocartesian category structure which is strict and category-like.

Let o, m be sets. One can check that $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ is category-like.

The following propositions are true:

(48) For every object a of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ holds $a = o$.

(49) For all objects a, b of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ holds $a = b$.

(50) For every morphism f of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ holds $f = m$.

(51) For all morphisms f, g of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ holds $f = g$.

(52) For all objects a, b of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ and for every morphism f of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ holds $f \in \mathrm{hom}(a,b)$.

(53) For all objects a, b of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ holds every morphism of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ is a morphism from a to b.

(54) For all objects a, b of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ holds $\mathrm{hom}(a,b) \neq \emptyset$.

(55) Every object of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ is initial.

(56) For every object c of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ and for all morphisms i_1, i_2 of $\circlearrowleft_{\mathrm{c}}^{\mathrm{op}}(o,m)$ holds c is a coproduct w.r.t. i_1 and i_2.

Let I_1 be a category-like cocartesian category structure. We say that I_1 is cocartesian if and only if the conditions (Def. 27) are satisfied.

(Def. 27)(i) The initial of I_1 is initial, and

(ii) for all objects a, b of I_1 holds dom(the 1st-coprojection of I_1)($\langle a, b \rangle$) $= a$ and dom(the 2nd-coprojection of I_1)($\langle a, b \rangle$) $= b$ and (the coproduct of I_1)($\langle a, b \rangle$) is a coproduct w.r.t. (the 1st-coprojection of I_1)($\langle a, b \rangle$) and (the 2nd-coprojection of I_1)($\langle a, b \rangle$).

We now state the proposition

(57) For all sets o, m holds $\circlearrowright_{\mathrm{c}}^{\mathrm{op}}(o,m)$ is cocartesian.

Let us observe that there exists a category-like cocartesian category structure which is strict and cocartesian.

A cocartesian category is a cocartesian category-like cocartesian category structure.

We adopt the following convention: C denotes a cocartesian category and a, b, c, d, e, s denote objects of C.

The following propositions are true:

(58) $\mathbf{0}_C$ is initial.

(59) For all morphisms f_1, f_2 from $\mathbf{0}_C$ to a holds $f_1 = f_2$.

Let us consider C, a.

(Def. 28) $\mathrm{init}\, a$ is a morphism from $\mathbf{0}_C$ to a.

The following propositions are true:

(60) $\hom(\mathbf{0}_C, a) \neq \emptyset$.

(61) $\mathrm{init}\, a = \mathrm{init}(\mathbf{0}_C, a)$.

(62) $\mathrm{dom}\,\mathrm{init}\, a = \mathbf{0}_C$ and $\mathrm{cod}\,\mathrm{init}\, a = a$.

(63) $\hom(\mathbf{0}_C, a) = \{\mathrm{init}\, a\}$.

(64) $\mathrm{dom}\,\mathrm{in}_1(a+b) = a$ and $\mathrm{cod}\,\mathrm{in}_1(a+b) = a+b$.

(65) $\mathrm{dom}\,\mathrm{in}_2(a+b) = b$ and $\mathrm{cod}\,\mathrm{in}_2(a+b) = a+b$.

(66) $\hom(a, a+b) \neq \emptyset$ and $\hom(b, a+b) \neq \emptyset$.

(67) $a+b$ is a coproduct w.r.t. $\mathrm{in}_1(a+b)$ and $\mathrm{in}_2(a+b)$.

(68) C has finite coproduct.

(69) If $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$, then $\mathrm{in}_1(a+b)$ is coretraction and $\mathrm{in}_2(a+b)$ is coretraction.

Let us consider C, a, b. Then $\mathrm{in}_1(a+b)$ is a morphism from a to $a+b$. Then $\mathrm{in}_2(a+b)$ is a morphism from b to $a+b$.

Let us consider C, a, b, c, let f be a morphism from a to c, and let g be a morphism from b to c. Let us assume that $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$. The functor $\langle f, g \rangle$ yielding a morphism from $a+b$ to c is defined as follows:

(Def. 29) $\langle f, g \rangle \cdot \mathrm{in}_1(a+b) = f$ and $\langle f, g \rangle \cdot \mathrm{in}_2(a+b) = g$.

We now state several propositions:

(70) If $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$, then $\hom(a+b, c) \neq \emptyset$.

(71) $\langle \mathrm{in}_1(a+b), \mathrm{in}_2(a+b) \rangle = \mathrm{id}_{a+b}$.

(72) Let f be a morphism from a to c, g be a morphism from b to c, and h be a morphism from c to d. If $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$, then $\langle h \cdot f, h \cdot g \rangle = h \cdot \langle f, g \rangle$.

(73) Let f, k be morphisms from a to c and g, h be morphisms from b to c. If $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and $\langle f,g\rangle = \langle k,h\rangle$, then $f = k$ and $g = h$.

(74) Let f be a morphism from a to c and g be a morphism from b to c. If $\hom(a,c) \neq \emptyset$ and if $\hom(b,c) \neq \emptyset$ and if f is epi or g is epi, then $\langle f,g\rangle$ is epi.

(75) a and $a + \mathbf{0}_C$ are isomorphic and a and $\mathbf{0}_C + a$ are isomorphic.

(76) $a+b$ and $b+a$ are isomorphic.

(77) $(a+b)+c$ and $a+(b+c)$ are isomorphic.

Let us consider C, a. The functor ∇_a yielding a morphism from $a+a$ to a is defined by:

(Def. 30) $\nabla_a = \langle \mathrm{id}_a, \mathrm{id}_a\rangle$.

Let us consider C, a, b, c, d, let f be a morphism from a to c, and let g be a morphism from b to d. The functor $f+g$ yields a morphism from $a+b$ to $c+d$ and is defined as follows:

(Def. 31) $f+g = \langle \mathrm{in}_1(c+d)\cdot f, \mathrm{in}_2(c+d)\cdot g\rangle$.

We now state several propositions:

(78) If $\hom(a,c) \neq \emptyset$ and $\hom(b,d) \neq \emptyset$, then $\hom(a+b,c+d) \neq \emptyset$.

(79) $\mathrm{id}_a + \mathrm{id}_b = \mathrm{id}_{a+b}$.

(80) Let f be a morphism from a to c, h be a morphism from b to d, g be a morphism from c to e, and k be a morphism from d to e. If $\hom(a,c) \neq \emptyset$ and $\hom(b,d) \neq \emptyset$ and $\hom(c,e) \neq \emptyset$ and $\hom(d,e) \neq \emptyset$, then $\langle g,k\rangle \cdot (f+h) = \langle g\cdot f, k\cdot h\rangle$.

(81) Let f be a morphism from a to c and g be a morphism from b to c. If $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$, then $\nabla_c \cdot (f+g) = \langle f,g\rangle$.

(82) Let f be a morphism from a to c, h be a morphism from b to d, g be a morphism from c to e, and k be a morphism from d to s. If $\hom(a,c) \neq \emptyset$ and $\hom(b,d) \neq \emptyset$ and $\hom(c,e) \neq \emptyset$ and $\hom(d,s) \neq \emptyset$, then $(g+k)\cdot(f+h) = g\cdot f + k\cdot h$.

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