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Subcategories and Products of Categories

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Summary. The *subcategory* of a category and product of categories is defined. The *inclusion functor* is the injection (inclusion) map $\overset{E}{\hookrightarrow}$ which sends each object and each arrow of a Subcategory E of a category C to itself (in C). The inclusion functor is faithful. *Full subcategories* of C, that is, those subcategories E of C such that $\mathrm{Hom}_E(a,b) = \mathrm{Hom}_C(b,b)$ for any objects a,b of E, are defined. A subcategory E of C is full when the inclusion functor $\overset{E}{\hookrightarrow}$ is full. The proposition that a full subcategory is determined by giving the set of objects of a category is proved. The product of two categories B and C is constructed in the usual way. Moreover, some simple facts on *bifunctors* (functors from a product category) are proved. The final notions in this article are that of projection functors and product of two functors (*complex* functors and *product* functors).

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The articles [10], [7], [12], [9], [13], [3], [4], [6], [2], [8], [1], [11], and [5] provide the notation and terminology for this paper.

For simplicity, we use the following convention: X is a set, C, D, E are non empty sets, c is an element of C, and d is an element of D.

Let us consider D, X, E, let F be a non empty set of functions from X to E, let f be a function from D into F, and let d be an element of D. Then $f(d)$ is an element of F.

In the sequel f is a function from $[:C, D:]$ into E.

One can prove the following two propositions:

(1) curry f is a function from C into E^D.

(2) curry′ f is a function from D into E^C.

Let us consider C, D, E, f. Then curry f is a function from C into E^D. Then curry′ f is a function from D into E^C.

One can prove the following two propositions:

(3) $f(\langle c, d\rangle) = (\mathrm{curry}\, f)(c)(d)$.

(4) $f(\langle c, d\rangle) = (\mathrm{curry}'\, f)(d)(c)$.

In the sequel B, C, D, C', D' are categories.

Let us consider B, C and let c be an object of C. The functor $B \longmapsto c$ yielding a functor from B to C is defined by:

(Def. 1) $B \longmapsto c = (\text{the morphisms of } B) \longmapsto \mathrm{id}_c$.

Next we state two propositions:

(6)[1] For every object c of C and for every morphism f of B holds $(B \longmapsto c)(f) = \mathrm{id}_c$.

[1] The proposition (5) has been removed.

(7) For every object c of C and for every object b of B holds $(\mathrm{Obj}(B \longmapsto c))(b) = c$.

Let us consider C, D. The functor $\mathrm{Funct}(C,D)$ yields a set and is defined as follows:

(Def. 2) For every set x holds $x \in \mathrm{Funct}(C,D)$ iff x is a functor from C to D.

Let us consider C, D. One can check that $\mathrm{Funct}(C,D)$ is non empty.

Let us consider C, D. A non empty set is called a non empty set of functors from C into D if:

(Def. 3) Every element of it is a functor from C to D.

Let us consider C, D and let F be a non empty set of functors from C into D. We see that the element of F is a functor from C to D.

Let A be a non empty set, let us consider C, D, let F be a non empty set of functors from C into D, let T be a function from A into F, and let x be an element of A. Then $T(x)$ is an element of F.

Let us consider C, D. Then $\mathrm{Funct}(C,D)$ is a non empty set of functors from C into D.

Let us consider C. A category is called a subcategory of C if it satisfies the conditions (Def. 4).

(Def. 4)(i) The objects of it $\subseteq$ the objects of C,

(ii) for all objects a, b of it and for all objects a', b' of C such that $a = a'$ and $b = b'$ holds $\hom(a,b) \subseteq \hom(a',b')$,

(iii) the composition of it $\leq$ the composition of C, and

(iv) for every object a of it and for every object a' of C such that $a = a'$ holds $\mathrm{id}_a = \mathrm{id}_{a'}$.

Let us consider C. One can verify that there exists a subcategory of C which is strict.

In the sequel E denotes a subcategory of C.

Next we state several propositions:

(12)[2] Every object of E is an object of C.

(13) The morphisms of $E \subseteq$ the morphisms of C.

(14) Every morphism of E is a morphism of C.

(15) For every morphism f of E and for every morphism f' of C such that $f = f'$ holds $\mathrm{dom}\, f = \mathrm{dom}\, f'$ and $\mathrm{cod}\, f = \mathrm{cod}\, f'$.

(16) Let a, b be objects of E, a', b' be objects of C, and f be a morphism from a to b. If $a = a'$ and $b = b'$ and $\hom(a,b) \neq \emptyset$, then f is a morphism from a' to b'.

(17) For all morphisms f, g of E and for all morphisms f', g' of C such that $f = f'$ and $g = g'$ and $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $g \cdot f = g' \cdot f'$.

(18) C is a subcategory of C.

(19) id_E is a functor from E to C.

Let us consider C, E. The functor $\overset{E}{\hookrightarrow}$ yielding a functor from E to C is defined as follows:

(Def. 5) $\overset{E}{\hookrightarrow} = \mathrm{id}_E$.

We now state several propositions:

(21)[3] For every morphism f of E holds $(\overset{E}{\hookrightarrow})(f) = f$.

(22) For every object a of E holds $(\mathrm{Obj}(\overset{E}{\hookrightarrow}))(a) = a$.

(23) For every object a of E holds $(\overset{E}{\hookrightarrow})(a) = a$.

[2] The propositions (8)–(11) have been removed.

[3] The proposition (20) has been removed.

(24) $\overset{E}{\hookrightarrow}$ is faithful.

(25) $\overset{E}{\hookrightarrow}$ is full if and only if for all objects a, b of E and for all objects a', b' of C such that $a = a'$ and $b = b'$ holds $\hom(a,b) = \hom(a',b')$.

Let C be a category structure and let us consider D. We say that C is full subcategory of D if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) C is a subcategory of D, and

(ii) for all objects a, b of C and for all objects a', b' of D such that $a = a'$ and $b = b'$ holds $\hom(a,b) = \hom(a',b')$.

The following propositions are true:

(27)[4] E is full subcategory of C iff $\overset{E}{\hookrightarrow}$ is full.

(28) Let O be a non empty subset of the objects of C. Then $\bigcup\{\hom(a,b); a$ ranges over objects of C, b ranges over objects of C: $a \in O \wedge b \in O\}$ is a non empty subset of the morphisms of C.

(29) Let O be a non empty subset of the objects of C and M be a non empty set. Suppose $M = \bigcup\{\hom(a,b); a$ ranges over objects of C, b ranges over objects of C: $a \in O \wedge b \in O\}$. Then

(i) (the dom-map of C)$\restriction M$ is a function from M into O,

(ii) (the cod-map of C)$\restriction M$ is a function from M into O,

(iii) (the composition of C)$\restriction [:M, M:]$ is a partial function from $[:M, M:]$ to M, and

(iv) (the id-map of C)$\restriction O$ is a function from O into M.

(30) Let O be a non empty subset of the objects of C, M be a non empty set, d, c be functions from M into O, p be a partial function from $[:M, M:]$ to M, and i be a function from O into M. Suppose that

(i) $M = \bigcup\{\hom(a,b); a$ ranges over objects of C, b ranges over objects of C: $a \in O \wedge b \in O\}$,

(ii) $d =$ (the dom-map of C)$\restriction M$,

(iii) $c =$ (the cod-map of C)$\restriction M$,

(iv) $p =$ (the composition of C)$\restriction [:M, M:]$, and

(v) $i =$ (the id-map of C)$\restriction O$.

Then $\langle O, M, d, c, p, i\rangle$ is full subcategory of C.

(31) Let O be a non empty subset of the objects of C, M be a non empty set, d, c be functions from M into O, p be a partial function from $[:M, M:]$ to M, and i be a function from O into M. Suppose $\langle O, M, d, c, p, i\rangle$ is full subcategory of C. Then

(i) $M = \bigcup\{\hom(a,b); a$ ranges over objects of C, b ranges over objects of C: $a \in O \wedge b \in O\}$,

(ii) $d =$ (the dom-map of C)$\restriction M$,

(iii) $c =$ (the cod-map of C)$\restriction M$,

(iv) $p =$ (the composition of C)$\restriction [:M, M:]$, and

(v) $i =$ (the id-map of C)$\restriction O$.

Let X_1, X_2, Y_1, Y_2 be non empty sets, let f_1 be a function from X_1 into Y_1, and let f_2 be a function from X_2 into Y_2. Then $[:f_1, f_2:]$ is a function from $[:X_1, X_2:]$ into $[:Y_1, Y_2:]$.

Let A, B be non empty sets, let f be a partial function from $[:A, A:]$ to A, and let g be a partial function from $[:B, B:]$ to B. Then $|:f, g:|$ is a partial function from $[:[:A, B:], [:A, B:]:]$ to $[:A, B:]$.

Let us consider C, D. The functor $[:C, D:]$ yields a category and is defined by the condition (Def. 7).

[4] The proposition (26) has been removed.

(Def. 7) $[:C, D:] = \langle [:$the objects of $C,$ the objects of $D:], [:$the morphisms of $C,$ the morphisms of $D:], [:$the dom-map of $C,$ the dom-map of $D:], [:$the cod-map of $C,$ the cod-map of $D:], |:$the composition of $C,$ the composition of $D:|, [:$the id-map of $C,$ the id-map of $D:]\rangle$.

Let us consider C, D. Note that $[:C, D:]$ is strict.

Next we state two propositions:

(33)[5] (i) The objects of $[:C, D:] = [:$the objects of $C,$ the objects of $D:]$,

(ii) the morphisms of $[:C, D:] = [:$the morphisms of $C,$ the morphisms of $D:]$,

(iii) the dom-map of $[:C, D:] = [:$the dom-map of $C,$ the dom-map of $D:]$,

(iv) the cod-map of $[:C, D:] = [:$the cod-map of $C,$ the cod-map of $D:]$,

(v) the composition of $[:C, D:] = |:$the composition of $C,$ the composition of $D:|$, and

(vi) the id-map of $[:C, D:] = [:$the id-map of $C,$ the id-map of $D:]$.

(34) For every object c of C and for every object d of D holds $\langle c, d\rangle$ is an object of $[:C, D:]$.

Let us consider C, D, let c be an object of C, and let d be an object of D. Then $\langle c, d\rangle$ is an object of $[:C, D:]$.

We now state two propositions:

(35) For every object c_1 of $[:C, D:]$ there exists an object c of C and there exists an object d of D such that $c_1 = \langle c, d\rangle$.

(36) For every morphism f of C and for every morphism g of D holds $\langle f, g\rangle$ is a morphism of $[:C, D:]$.

Let us consider C, D, let f be a morphism of C, and let g be a morphism of D. Then $\langle f, g\rangle$ is a morphism of $[:C, D:]$.

The following propositions are true:

(37) For every morphism f_3 of $[:C, D:]$ there exists a morphism f of C and there exists a morphism g of D such that $f_3 = \langle f, g\rangle$.

(38) For every morphism f of C and for every morphism g of D holds $\mathrm{dom}\langle f, g\rangle = \langle \mathrm{dom} f, \mathrm{dom} g\rangle$ and $\mathrm{cod}\langle f, g\rangle = \langle \mathrm{cod} f, \mathrm{cod} g\rangle$.

(39) For all morphisms f, f' of C and for all morphisms g, g' of D such that $\mathrm{dom} f' = \mathrm{cod} f$ and $\mathrm{dom} g' = \mathrm{cod} g$ holds $\langle f', g'\rangle \cdot \langle f, g\rangle = \langle f' \cdot f, g' \cdot g\rangle$.

(40) For all morphisms f, f' of C and for all morphisms g, g' of D such that $\mathrm{dom}\langle f', g'\rangle = \mathrm{cod}\langle f, g\rangle$ holds $\langle f', g'\rangle \cdot \langle f, g\rangle = \langle f' \cdot f, g' \cdot g\rangle$.

(41) For every object c of C and for every object d of D holds $\mathrm{id}_{\langle c, d\rangle} = \langle \mathrm{id}_c, \mathrm{id}_d\rangle$.

(42) For all objects c, c' of C and for all objects d, d' of D holds $\mathrm{hom}(\langle c, d\rangle, \langle c', d'\rangle) = [:\mathrm{hom}(c, c'), \mathrm{hom}(d, d'):]$.

(43) Let c, c' be objects of C, f be a morphism from c to c', d, d' be objects of D, and g be a morphism from d to d'. If $\mathrm{hom}(c, c') \neq \emptyset$ and $\mathrm{hom}(d, d') \neq \emptyset$, then $\langle f, g\rangle$ is a morphism from $\langle c, d\rangle$ to $\langle c', d'\rangle$.

(44) For every functor S from $[:C, C':]$ to D and for every object c of C holds $(\mathrm{curry}\, S)(\mathrm{id}_c)$ is a functor from C' to D.

(45) For every functor S from $[:C, C':]$ to D and for every object c' of C' holds $(\mathrm{curry}'\, S)(\mathrm{id}_{c'})$ is a functor from C to D.

[5] The proposition (32) has been removed.

Let us consider C, C', D, let S be a functor from $[:C, C':]$ to D, and let c be an object of C. The functor $S(c,-)$ yields a functor from C' to D and is defined as follows:

(Def. 8) $S(c,-) = (\mathrm{curry}\,S)(\mathrm{id}_c)$.

Next we state two propositions:

(47)[6] For every functor S from $[:C, C':]$ to D and for every object c of C and for every morphism f of C' holds $S(c,-)(f) = S(\langle \mathrm{id}_c, f \rangle)$.

(48) For every functor S from $[:C, C':]$ to D and for every object c of C and for every object c' of C' holds $(\mathrm{Obj}(S(c,-)))(c') = (\mathrm{Obj}\,S)(\langle c, c' \rangle)$.

Let us consider C, C', D, let S be a functor from $[:C, C':]$ to D, and let c' be an object of C'. The functor $S(-,c')$ yielding a functor from C to D is defined as follows:

(Def. 9) $S(-,c') = (\mathrm{curry}'\,S)(\mathrm{id}_{c'})$.

One can prove the following propositions:

(50)[7] For every functor S from $[:C, C':]$ to D and for every object c' of C' and for every morphism f of C holds $S(-,c')(f) = S(\langle f, \mathrm{id}_{c'} \rangle)$.

(51) For every functor S from $[:C, C':]$ to D and for every object c of C and for every object c' of C' holds $(\mathrm{Obj}(S(-,c')))(c) = (\mathrm{Obj}\,S)(\langle c, c' \rangle)$.

(52) Let L be a function from the objects of C into $\mathrm{Funct}(B,D)$ and M be a function from the objects of B into $\mathrm{Funct}(C,D)$. Suppose that

(i) for every object c of C and for every object b of B holds $M(b)(\mathrm{id}_c) = L(c)(\mathrm{id}_b)$, and

(ii) for every morphism f of B and for every morphism g of C holds $M(\mathrm{cod}\,f)(g) \cdot L(\mathrm{dom}\,g)(f) = L(\mathrm{cod}\,g)(f) \cdot M(\mathrm{dom}\,f)(g)$.

Then there exists a functor S from $[:B, C:]$ to D such that for every morphism f of B and for every morphism g of C holds $S(\langle f, g \rangle) = L(\mathrm{cod}\,g)(f) \cdot M(\mathrm{dom}\,f)(g)$.

(53) Let L be a function from the objects of C into $\mathrm{Funct}(B,D)$ and M be a function from the objects of B into $\mathrm{Funct}(C,D)$. Given a functor S from $[:B, C:]$ to D such that let c be an object of C and b be an object of B. Then $S(-,c) = L(c)$ and $S(b,-) = M(b)$. Let f be a morphism of B and g be a morphism of C. Then $M(\mathrm{cod}\,f)(g) \cdot L(\mathrm{dom}\,g)(f) = L(\mathrm{cod}\,g)(f) \cdot M(\mathrm{dom}\,f)(g)$.

(54) π_1((the morphisms of C) $\times$ the morphisms of D) is a functor from $[:C, D:]$ to C.

(55) π_2((the morphisms of C) $\times$ the morphisms of D) is a functor from $[:C, D:]$ to D.

Let us consider C, D. The functor $\pi_1(C \times D)$ yielding a functor from $[:C, D:]$ to C is defined as follows:

(Def. 10) $\pi_1(C \times D) = \pi_1$((the morphisms of C) $\times$ the morphisms of D).

The functor $\pi_2(C \times D)$ yields a functor from $[:C, D:]$ to D and is defined as follows:

(Def. 11) $\pi_2(C \times D) = \pi_2$((the morphisms of C) $\times$ the morphisms of D).

The following propositions are true:

(58)[8] For every morphism f of C and for every morphism g of D holds $\pi_1(C \times D)(\langle f, g \rangle) = f$.

(59) For every object c of C and for every object d of D holds $(\mathrm{Obj}\,\pi_1(C \times D))(\langle c, d \rangle) = c$.

(60) For every morphism f of C and for every morphism g of D holds $\pi_2(C \times D)(\langle f, g \rangle) = g$.

[6] The proposition (46) has been removed.
[7] The proposition (49) has been removed.
[8] The propositions (56) and (57) have been removed.

(61) For every object c of C and for every object d of D holds $(\mathrm{Obj}\,\pi_2(C \times D))(\langle c, d \rangle) = d$.

(62) For every functor T from C to D and for every functor T' from C to D' holds $\langle T, T' \rangle$ is a functor from C to $[: D, D' :]$.

Let us consider C, D, D', let T be a functor from C to D, and let T' be a functor from C to D'. Then $\langle T, T' \rangle$ is a functor from C to $[: D, D' :]$.

One can prove the following three propositions:

(63) Let T be a functor from C to D, T' be a functor from C to D', and c be an object of C. Then $(\mathrm{Obj}\langle T, T' \rangle)(c) = \langle (\mathrm{Obj}\,T)(c), (\mathrm{Obj}\,T')(c) \rangle$.

(64) For every functor T from C to D and for every functor T' from C' to D' holds $[: T, T' :] = \langle T \cdot \pi_1(C \times C'), T' \cdot \pi_2(C \times C') \rangle$.

(65) For every functor T from C to D and for every functor T' from C' to D' holds $[: T, T' :]$ is a functor from $[: C, C' :]$ to $[: D, D' :]$.

Let us consider C, C', D, D', let T be a functor from C to D, and let T' be a functor from C' to D'. Then $[: T, T' :]$ is a functor from $[: C, C' :]$ to $[: D, D' :]$.

Next we state the proposition

(66) Let T be a functor from C to D, T' be a functor from C' to D', c be an object of C, and c' be an object of C'. Then $(\mathrm{Obj}[: T, T' :])(\langle c, c' \rangle) = \langle (\mathrm{Obj}\,T)(c), (\mathrm{Obj}\,T')(c') \rangle$.

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