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Mahlo and Inaccessible Cardinals

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Summary. This article contains basic ordinal topology: closed unbounded and stationary sets and necessary theorems about them, completness of the centered system of Clubs of M, Mahlo and strongly Mahlo cardinals, the proof that (strongly) Mahlo is (strongly) inaccessible, and the proof that Rank of strongly inaccessible is a model of ZF.

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The articles [15], [11], [17], [16], [19], [7], [9], [14], [12], [3], [10], [4], [1], [6], [8], [18], [5], [13], and [2] provide the notation and terminology for this paper.

1. Clubs and Mahlo Cardinals

Let S be a set, let X be a set, and let Y be a subset of S. Then $X \cap Y$ is a subset of S.

Let us note that every ordinal number which is cardinal and infinite is also limit.

Let us observe that every ordinal number which is non empty and limit is also infinite.

One can check that every aleph which is non limit is also non countable.

Let us observe that there exists an aleph which is regular and non countable.

We adopt the following convention: A, B denote limit infinite ordinal numbers, B_1, B_2, B_3, C denote ordinal numbers, and X denotes a set.

Let us consider A, X. We say that X is unbounded in A if and only if:

(Def. 1) $X \subseteq A$ and $\sup X = A$.

We say that X is closed in A if and only if:

(Def. 2) $X \subseteq A$ and for every B such that $B \in A$ holds if $\sup(X \cap B) = B$, then $B \in X$.

Let us consider A, X. We say that X is club in A if and only if:

(Def. 3) X is closed in A and X is unbounded in A.

In the sequel X denotes a subset of A.

Let us consider A, X. We say that X is unbounded if and only if:

(Def. 4) $\sup X = A$.

We introduce X is bounded as an antonym of X is unbounded. We say that X is closed if and only if:

(Def. 5) For every B such that $B \in A$ holds if $\sup(X \cap B) = B$, then $B \in X$.

The following propositions are true:

(2)[1] X is club in A iff X is closed and unbounded.

(3) $X \subseteq \sup X$.

(4) Suppose X is non empty and for every B_1 such that $B_1 \in X$ there exists B_2 such that $B_2 \in X$ and $B_1 \in B_2$. Then $\sup X$ is limit infinite ordinal number.

(5) X is bounded iff there exists B_1 such that $B_1 \in A$ and $X \subseteq B_1$.

(6) If $\sup(X \cap B) \neq B$, then there exists B_1 such that $B_1 \in B$ and $X \cap B \subseteq B_1$.

(7) X is unbounded iff for every B_1 such that $B_1 \in A$ there exists C such that $C \in X$ and $B_1 \subseteq C$.

(8) If X is unbounded, then X is non empty.

(9) If X is unbounded and $B_1 \in A$, then there exists an element B_3 of A such that $B_3 \in \{B_2; B_2$ ranges over elements of A: $B_2 \in X \ \wedge \ B_1 \in B_2\}$.

Let us consider A, X, B_1. Let us assume that X is unbounded. And let us assume that $B_1 \in A$. The functor $\mathrm{LBound}(B_1, X)$ yields an element of X and is defined by:

(Def. 6) $\mathrm{LBound}(B_1, X) = \inf\{B_2; B_2$ ranges over elements of A: $B_2 \in X \ \wedge \ B_1 \in B_2\}$.

The following propositions are true:

(10) If X is unbounded and $B_1 \in A$, then $\mathrm{LBound}(B_1, X) \in X$ and $B_1 \in \mathrm{LBound}(B_1, X)$.

(11) Ω_A is closed and unbounded.

Let A be a set, let X be a subset of A, and let Y be a set. Then $X \setminus Y$ is a subset of A.
We now state two propositions:

(12) If $B_1 \in A$ and X is closed and unbounded, then $X \setminus B_1$ is closed and unbounded.

(13) If $B_1 \in A$, then $A \setminus B_1$ is closed and unbounded.

Let us consider A, X. We say that X is stationary if and only if:

(Def. 7) For every subset Y of A such that Y is closed and unbounded holds $X \cap Y$ is non empty.

Next we state the proposition

(14) For all subsets X, Y of A such that X is stationary and $X \subseteq Y$ holds Y is stationary.

Let us consider A and let X be a set. We say that X is stationary in A if and only if:

(Def. 8) $X \subseteq A$ and for every subset Y of A such that Y is closed and unbounded holds $X \cap Y$ is non empty.

One can prove the following proposition

(15) For all sets X, Y such that X is stationary in A and $X \subseteq Y$ and $Y \subseteq A$ holds Y is stationary in A.

Let X be a set and let S be a family of subsets of X. We see that the element of S is a subset of X.
The following proposition is true

(16) If X is stationary, then X is unbounded.

Let us consider A, X. The functor $\mathrm{limpoints} X$ yielding a subset of A is defined by:

[1] The proposition (1) has been removed.

(Def. 9) $\operatorname{limpoints} X = \{B_1; B_1$ ranges over elements of A: B_1 is infinite and limit $\wedge$ $\sup(X \cap B_1) = B_1\}$.

The following four propositions are true:

(17) If $X \cap B_3 \subseteq B_1$, then $B_3 \cap \operatorname{limpoints} X \subseteq \operatorname{succ} B_1$.

(18) If $X \subseteq B_1$, then $\operatorname{limpoints} X \subseteq \operatorname{succ} B_1$.

(19) $\operatorname{limpoints} X$ is closed.

(20) Suppose X is unbounded and $\operatorname{limpoints} X$ is bounded. Then there exists B_1 such that $B_1 \in A$ and $\{\operatorname{succ} B_2; B_2$ ranges over elements of A: $B_2 \in X \wedge B_1 \in \operatorname{succ} B_2\}$ is club in A.

In the sequel M is a non countable aleph and X is a subset of M.

Let us consider M. One can check that there exists an element of M which is cardinal and infinite.

In the sequel N denotes a cardinal infinite element of M.

Next we state several propositions:

(21) For every aleph M and for every subset X of M such that X is unbounded holds $\operatorname{cf} M \leq \overline{\overline{X}}$.

(22) For every family S of subsets of M such that every element of S is closed holds $\bigcap S$ is closed.

(23) If $\aleph_0 < \operatorname{cf} M$, then for every function f from $\mathbb{N}$ into X holds $\sup \operatorname{rng} f \in M$.

(24) Suppose $\aleph_0 < \operatorname{cf} M$. Let S be a non empty family of subsets of M. If $\overline{\overline{S}} < \operatorname{cf} M$ and every element of S is closed and unbounded, then $\bigcap S$ is closed and unbounded.

(25) If $\aleph_0 < \operatorname{cf} M$ and X is unbounded, then for every B_1 such that $B_1 \in M$ there exists B such that $B \in M$ and $B_1 \in B$ and $B \in \operatorname{limpoints} X$.

(26) If $\aleph_0 < \operatorname{cf} M$ and X is unbounded, then $\operatorname{limpoints} X$ is unbounded.

Let us consider M. We say that M is Mahlo if and only if:

(Def. 10) $\{N : N \text{is regular}\}$ is stationary in M.

We introduce M is Mahlo as a synonym of M is Mahlo. We say that M is strongly Mahlo if and only if:

(Def. 11) $\{N : N \text{is strongly inaccessible}\}$ is stationary in M.

We introduce M is strongly Mahlo as a synonym of M is strongly Mahlo.

One can prove the following propositions:

(27) If M is strongly Mahlo, then M is Mahlo.

(28) If M is Mahlo, then M is regular.

(29) If M is Mahlo, then M is limit.

(30) If M is Mahlo, then M is inaccessible.

(31) If M is strongly Mahlo, then M is strong limit.

(32) If M is strongly Mahlo, then M is strongly inaccessible.

2. Proof that Strongly Inaccessible is Model of ZF

We follow the rules: A is an ordinal number, x, y are sets, and X, Y are sets.

Next we state several propositions:

(33) Suppose that for every x such that $x \in X$ there exists y such that $y \in X$ and $x \subseteq y$ and y is a cardinal number. Then $\bigcup X$ is a cardinal number.

(34) For every aleph M such that $\overline{\overline{X}} < \mathrm{cf}\, M$ and for every Y such that $Y \in X$ holds $\overline{\overline{Y}} < M$ holds $\overline{\overline{\bigcup X}} \in M$.

(35) If M is strongly inaccessible and $A \in M$, then $\overline{\overline{\mathbf{R}_A}} < M$.

(36) If M is strongly inaccessible, then $\overline{\overline{\mathbf{R}_M}} = M$.

(37) If M is strongly inaccessible, then $\mathbf{R}_M$ is a Tarski class.

(38) For every non empty ordinal number A holds $\mathbf{R}_A$ is non empty.

Let A be a non empty ordinal number. Note that $\mathbf{R}_A$ is non empty.

We now state two propositions:

(39) If M is strongly inaccessible, then $\mathbf{R}_M$ is a universal class.

(40) If M is strongly inaccessible, then $\mathbf{R}_M$ is model of ZF.

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