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The Cantor Set[1]

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Summary. The aim of the paper is to define some basic notions of the theory of topological spaces like basis and prebasis, and to prove their simple properties. The definition of the Cantor set is given in terms of countable product of $\{0,1\}$ and a collection of its subsets to serve as a prebasis.

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The articles [10], [4], [12], [11], [7], [13], [2], [3], [6], [8], [5], [1], and [9] provide the notation and terminology for this paper.

Let Y be a set and let x be a non empty set. Note that $Y \longmapsto x$ is non-empty.

Let X be a set and let A be a family of subsets of X. The functor $\mathrm{UniCl}(A)$ yields a family of subsets of X and is defined as follows:

(Def. 1) For every subset x of X holds $x \in \mathrm{UniCl}(A)$ iff there exists a family Y of subsets of X such that $Y \subseteq A$ and $x = \bigcup Y$.

Let X be a topological structure. A family of subsets of X is said to be a basis of X if:

(Def. 2) It $\subseteq$ the topology of X and the topology of $X \subseteq \mathrm{UniCl}(\mathrm{it})$.

One can prove the following propositions:

(1) For every set X and for every family A of subsets of X holds $A \subseteq \mathrm{UniCl}(A)$.

(2) For every topological structure S holds the topology of S is a basis of S.

(3) For every topological structure S holds the topology of S is open.

Let M be a set and let B be a family of subsets of M. The functor $\mathrm{Intersect}(B)$ yielding a subset of M is defined as follows:

(Def. 3) $\mathrm{Intersect}(B) = \begin{cases} \bigcap B, & \text{if } B \neq \emptyset, \\ M, & \text{otherwise.} \end{cases}$

Let X be a set and let A be a family of subsets of X. The functor $\mathrm{FinMeetCl}(A)$ yields a family of subsets of X and is defined by the condition (Def. 4).

(Def. 4) Let x be a subset of X. Then $x \in \mathrm{FinMeetCl}(A)$ if and only if there exists a family Y of subsets of X such that $Y \subseteq A$ and Y is finite and $x = \mathrm{Intersect}(Y)$.

We now state the proposition

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(4) For every set X and for every family A of subsets of X holds $A \subseteq \mathrm{FinMeetCl}(A)$.

Let T be a non empty topological space. One can verify that the topology of T is non empty.
Next we state several propositions:

(5) For every non empty topological space T holds the topology of $T = \mathrm{FinMeetCl}($the topology of $T)$.

(6) For every topological space T holds the topology of $T = \mathrm{UniCl}($the topology of $T)$.

(7) For every non empty topological space T holds the topology of $T = \mathrm{UniCl}(\mathrm{FinMeetCl}($the topology of $T))$.

(8) For every set X and for every family A of subsets of X holds $X \in \mathrm{FinMeetCl}(A)$.

(9) For every set X and for all families A, B of subsets of X such that $A \subseteq B$ holds $\mathrm{UniCl}(A) \subseteq \mathrm{UniCl}(B)$.

(10) Let X be a set, R be a family of subsets of X, and x be a set. Suppose $x \in X$. Then $x \in \mathrm{Intersect}(R)$ if and only if for every set Y such that $Y \in R$ holds $x \in Y$.

(11) For every set X and for all families H, J of subsets of X such that $H \subseteq J$ holds $\mathrm{Intersect}(J) \subseteq \mathrm{Intersect}(H)$.

Let X be a set and let R be a family of subsets of 2^X. We see that the element of R is a family of subsets of X. Then $\bigcup R$ is a family of subsets of X.

One can prove the following proposition

(12) Let X be a set, R be a non empty family of subsets of 2^X, and F be a family of subsets of X. If $F = \{\mathrm{Intersect}(x) : x$ ranges over elements of $R\}$, then $\mathrm{Intersect}(F) = \mathrm{Intersect}(\bigcup R)$.

Let X, Y be sets, let A be a family of subsets of X, let F be a function from Y into 2^A, and let x be a set. Then $F(x)$ is a family of subsets of X.

We now state four propositions:

(13) For every set X and for every family A of subsets of X holds $\mathrm{FinMeetCl}(A) = \mathrm{FinMeetCl}(\mathrm{FinMeetCl}(A))$.

(14) For every set X and for every family A of subsets of X and for all sets a, b such that $a \in \mathrm{FinMeetCl}(A)$ and $b \in \mathrm{FinMeetCl}(A)$ holds $a \cap b \in \mathrm{FinMeetCl}(A)$.

(15) For every set X and for every family A of subsets of X and for all sets a, b such that $a \subseteq \mathrm{FinMeetCl}(A)$ and $b \subseteq \mathrm{FinMeetCl}(A)$ holds $a \Cap b \subseteq \mathrm{FinMeetCl}(A)$.

(16) For every set X and for all families A, B of subsets of X such that $A \subseteq B$ holds $\mathrm{FinMeetCl}(A) \subseteq \mathrm{FinMeetCl}(B)$.

Let X be a set and let A be a family of subsets of X. One can check that $\mathrm{FinMeetCl}(A)$ is non empty.

Next we state the proposition

(17) For every non empty set X and for every family A of subsets of X holds $\langle X, \mathrm{UniCl}(\mathrm{FinMeetCl}(A))\rangle$ is topological space-like.

Let X be a topological structure. A family of subsets of X is said to be a prebasis of X if:

(Def. 5) It $\subseteq$ the topology of X and there exists a basis F of X such that $F \subseteq \mathrm{FinMeetCl}(\mathrm{it})$.

One can prove the following propositions:

(18) For every non empty set X holds every family Y of subsets of X is a basis of $\langle X, \mathrm{UniCl}(Y)\rangle$.

(19) Let T_1, T_2 be strict non empty topological spaces and P be a prebasis of T_1. Suppose the carrier of $T_1 =$ the carrier of T_2 and P is a prebasis of T_2. Then $T_1 = T_2$.

(20) For every non empty set X holds every family Y of subsets of X is a prebasis of $\langle X, \mathrm{UniCl}(\mathrm{FinMeetCl}(Y))\rangle$.

The strict non empty topological space the Cantor set is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of the Cantor set $= \prod(\mathbb{N} \longmapsto \{0,1\})$, and

(ii) there exists a prebasis P of the Cantor set such that for every subset X of $\prod(\mathbb{N} \longmapsto \{0,1\})$ holds $X \in P$ iff there exist natural numbers N, n such that for every element F of $\prod(\mathbb{N} \longmapsto \{0,1\})$ holds $F \in X$ iff $F(N) = n$.

REFERENCES

[1] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/card_3.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[4] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[5] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/tops_2.html.

[6] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[7] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[8] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[9] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[11] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[13] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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