JOURNAL OF FORMALIZED MATHEMATICS
Volume 14, Released 2002, Published 2003
Inst. of Computer Science, Univ. of Białystok

On the Decompositions of Intervals and Simple Closed Curves[1]

Adam Grabowski
University of Białystok

Summary. The aim of the paper is to show that the only subcontinua of the Jordan curve are arcs, the whole curve, and singletons of its points. Additionally, it has been shown that the only subcontinua of the unit interval $\mathbb{I}$ are closed intervals.

MML Identifier: BORSUK_4.

WWW: http://mizar.org/JFM/Vol14/borsuk_4.html

The articles [23], [26], [1], [25], [27], [3], [5], [2], [4], [21], [8], [6], [22], [10], [12], [13], [17], [28], [20], [24], [15], [16], [18], [19], [11], [7], [14], and [9] provide the notation and terminology for this paper.

1. Preliminaries

One can check that every simple closed curve is non trivial.

Let T be a non empty topological space. One can check that there exists a subset of T which is non empty, compact, and connected.

Let us observe that every element of $\mathbb{I}$ is real.

One can prove the following two propositions:

(1) Let X be a non empty set and A, B be non empty subsets of X. If $A \subset B$, then there exists an element p of X such that $p \in B$ and $A \subseteq B \setminus \{p\}$.

(2) Let X be a non empty set and A be a non empty subset of X. Then A is trivial if and only if there exists an element x of X such that $A = \{x\}$.

Let T be a non trivial 1-sorted structure. Note that there exists a subset of T which is non trivial. The following proposition is true

(3) For every non trivial set X and for every set p there exists an element q of X such that $q \neq p$.

Let X be a non trivial set. One can verify that there exists a subset of X which is non trivial. We now state a number of propositions:

(4) Let T be a non trivial set, X be a non trivial subset of T, and p be a set. Then there exists an element q of T such that $q \in X$ and $q \neq p$.

[1]This work has been partially supported by the European Community TYPES grant IST-1999-29001 and CALCULEMUS grant HPRN-CT-2000-00102.

(5) Let f, g be functions and a be a set. Suppose f is one-to-one and g is one-to-one and $\operatorname{dom} f \cap \operatorname{dom} g = \{a\}$ and $\operatorname{rng} f \cap \operatorname{rng} g = \{f(a)\}$. Then $f{+\cdot}g$ is one-to-one.

(6) Let f, g be functions and a be a set. Suppose f is one-to-one and g is one-to-one and $\operatorname{dom} f \cap \operatorname{dom} g = \{a\}$ and $\operatorname{rng} f \cap \operatorname{rng} g = \{f(a)\}$ and $f(a) = g(a)$. Then $(f{+\cdot}g)^{-1} = f^{-1}{+\cdot}g^{-1}$.

(7) Let n be a natural number, A be a non empty subset of $\mathcal{E}^n_{\rm T}$, and p, q be points of $\mathcal{E}^n_{\rm T}$. If A is an arc from p to q, then $A \setminus \{p\}$ is non empty.

(8) For every natural number n and for all points a, b of $\mathcal{E}^n_{\rm T}$ holds $\mathcal{L}(a,b)$ is convex.

(9) For all real numbers s_1, s_3, s_4, l such that $s_1 \leq s_3$ and $s_1 < s_4$ and $0 \leq l$ and $l \leq 1$ holds $s_1 \leq (1-l) \cdot s_3 + l \cdot s_4$.

(10) For every set x and for all real numbers a, b such that $a \leq b$ and $x \in [a,b]$ holds $x \in]a,b[$ or $x = a$ or $x = b$.

(11) For all real numbers a, b, c, d such that $]a,b[$ meets $[c,d]$ holds $b > c$.

(12) For all real numbers a, b, c, d such that $b \leq c$ holds $[a,b]$ misses $]c,d[$.

(13) For all real numbers a, b, c, d such that $b \leq c$ holds $]a,b[$ misses $[c,d]$.

(14) For all real numbers a, b, c, d such that $a < b$ and $[a,b] \subseteq [c,d]$ holds $c \leq a$ and $b \leq d$.

(15) For all real numbers a, b, c, d such that $a < b$ and $]a,b[\subseteq [c,d]$ holds $c \leq a$ and $b \leq d$.

(16) For all real numbers a, b, c, d such that $a < b$ and $]a,b[\subseteq [c,d]$ holds $[a,b] \subseteq [c,d]$.

(17) For every subset A of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A =]a,b[$ holds $[a,b] \subseteq$ the carrier of $\mathbb{I}$.

(18) For every subset A of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A =]a,b]$ holds $[a,b] \subseteq$ the carrier of $\mathbb{I}$.

(19) For every subset A of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A = [a,b[$ holds $[a,b] \subseteq$ the carrier of $\mathbb{I}$.

(20) For all real numbers a, b such that $a \neq b$ holds $\overline{]a,b]} = [a,b]$.

(21) For all real numbers a, b such that $a \neq b$ holds $\overline{[a,b[} = [a,b]$.

(22) For every subset A of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A =]a,b[$ holds $\overline{A} = [a,b]$.

(23) For every subset A of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A =]a,b]$ holds $\overline{A} = [a,b]$.

(24) For every subset A of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A = [a,b[$ holds $\overline{A} = [a,b]$.

(25) For all real numbers a, b such that $a < b$ holds $[a,b] \neq]a,b]$.

(26) For all real numbers a, b holds $[a,b[$ misses $\{b\}$ and $]a,b]$ misses $\{a\}$.

(27) For all real numbers a, b such that $a \leq b$ holds $[a,b] \setminus \{a\} =]a,b]$.

(28) For all real numbers a, b such that $a \leq b$ holds $[a,b] \setminus \{b\} = [a,b[$.

(29) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a,b] \cap [b,c[= \{b\}$.

(30) For all real numbers a, b, c holds $[a,b[$ misses $[b,c]$ and $[a,b]$ misses $]b,c]$.

(31) For all real numbers a, b, c such that $a \leq b$ and $b \leq c$ holds $[a,c] \setminus \{b\} = [a,b[\cup]b,c]$.

(32) For every subset A of $\mathbb{I}$ and for all real numbers a, b such that $a \leq b$ and $A = [a,b]$ holds $0 \leq a$ and $b \leq 1$.

(33) Let A, B be subsets of $\mathbb{I}$ and a, b, c be real numbers. If $a < b$ and $b < c$ and $A = [a,b[$ and $B =]b,c]$, then A and B are separated.

(34) For all real numbers a, b such that $a \leq b$ holds $[a,b] = [a,b[\cup \{b\}$.

(35) For all real numbers a, b such that $a \leq b$ holds $[a,b] = \{a\} \cup]a,b]$.

(36) For all real numbers a, b, c, d such that $a \leq b$ and $b < c$ and $c \leq d$ holds $[a,d] = [a,b] \cup]b,c[\cup [c,d]$.

(37) For all real numbers a, b, c, d such that $a \leq b$ and $b < c$ and $c \leq d$ holds $[a,d] \setminus ([a,b] \cup [c,d]) =]b,c[$.

(38) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a,b] \cup]b,c[=]a,c[$.

(39) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $[b,c[\subseteq]a,c[$.

(40) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a,b] \cup [b,c[=]a,c[$.

(41) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a,c[\setminus]a,b] =]b,c[$.

(42) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a,c[\setminus [b,c[=]a,b[$.

(43) For all points p_1, p_2 of $\mathbb{I}$ holds $[p_1, p_2]$ is a subset of $\mathbb{I}$.

(44) For all points a, b of $\mathbb{I}$ holds $]a,b[$ is a subset of $\mathbb{I}$.

2. Decompositions of Intervals

Next we state a number of propositions:

(45) For every real number p holds $\{p\}$ is a closed-interval subset of $\mathbb{R}$.

(46) Let A be a non empty connected subset of $\mathbb{I}$ and a, b, c be points of $\mathbb{I}$. If $a \leq b$ and $b \leq c$ and $a \in A$ and $c \in A$, then $b \in A$.

(47) For every non empty connected subset A of $\mathbb{I}$ and for all real numbers a, b such that $a \in A$ and $b \in A$ holds $[a,b] \subseteq A$.

(48) For all real numbers a, b and for every subset A of $\mathbb{I}$ such that $a \leq b$ and $A = [a,b]$ holds A is closed.

(49) For all points p_1, p_2 of $\mathbb{I}$ such that $p_1 \leq p_2$ holds $[p_1, p_2]$ is a non empty compact connected subset of $\mathbb{I}$.

(50) For every subset X of $\mathbb{I}$ and for every subset X' of $\mathbb{R}$ such that $X' = X$ holds X' is upper bounded and lower bounded.

(51) Let X be a subset of $\mathbb{I}$, X' be a subset of $\mathbb{R}$, and x be a real number. If $x \in X'$ and $X' = X$, then $\inf X' \leq x$ and $x \leq \sup X'$.

(52) For every subset A of $\mathbb{R}$ and for every subset B of $\mathbb{I}$ such that $A = B$ holds A is closed iff B is closed.

(53) For every closed-interval subset C of $\mathbb{R}$ holds $\inf C \leq \sup C$.

(54) Let C be a non empty compact connected subset of $\mathbb{I}$ and C' be a subset of $\mathbb{R}$. If $C = C'$ and $[\inf C', \sup C'] \subseteq C'$, then $[\inf C', \sup C'] = C'$.

(55) Every non empty compact connected subset of $\mathbb{I}$ is a closed-interval subset of $\mathbb{R}$.

(56) For every non empty compact connected subset C of $\mathbb{I}$ there exist points p_1, p_2 of $\mathbb{I}$ such that $p_1 \leq p_2$ and $C = [p_1, p_2]$.

3. Decompositions of Simple Closed Curves

The strict non empty subspace $I(01)$ of $\mathbb{I}$ is defined as follows:

(Def. 1) The carrier of $I(01) =]0,1[$.

One can prove the following propositions:

(57) For every subset A of $\mathbb{I}$ such that $A =$ the carrier of $I(01)$ holds $I(01) = \mathbb{I}\restriction A$.

(58) The carrier of $I(01) =$ (the carrier of $\mathbb{I}$) $\setminus \{0,1\}$.

(59) $I(01)$ is an open subspace of $\mathbb{I}$.

(60) For every real number r holds $r \in$ the carrier of $I(01)$ iff $0 < r$ and $r < 1$.

(61) For all points a, b of $\mathbb{I}$ such that $a < b$ and $b \neq 1$ holds $]a,b]$ is a non empty subset of $I(01)$.

(62) For all points a, b of $\mathbb{I}$ such that $a < b$ and $a \neq 0$ holds $[a,b[$ is a non empty subset of $I(01)$.

(63) For every simple closed curve D holds $(\mathcal{E}^2_{\mathrm{T}})\restriction\square_{\mathcal{E}^2}$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ are homeomorphic.

(64) Let D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If D is an arc from p_1 to p_2, then $I(01)$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction(D \setminus \{p_1,p_2\})$ are homeomorphic.

(65) Let D be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If D is an arc from p_1 to p_2, then $\mathbb{I}$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ are homeomorphic.

(66) For all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ holds $\mathbb{I}$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction\mathcal{L}(p_1,p_2)$ are homeomorphic.

(67) Let E be a subset of $I(01)$. Given points p_1, p_2 of $\mathbb{I}$ such that $p_1 < p_2$ and $E = [p_1,p_2]$. Then $\mathbb{I}$ and $I(01)\restriction E$ are homeomorphic.

(68) Let A be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and a, b be points of $\mathbb{I}$. Suppose A is an arc from p to q and $a < b$. Then there exists a non empty subset E of $\mathbb{I}$ and there exists a map f from $\mathbb{I}\restriction E$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction A$ such that $E = [a,b]$ and f is a homeomorphism and $f(a) = p$ and $f(b) = q$.

(69) Let A be a topological space, B be a non empty topological space, f be a map from A into B, C be a topological space, and X be a subset of A. Suppose f is continuous and C is a subspace of B. Let h be a map from $A\restriction X$ into C. If $h = f\restriction X$, then h is continuous.

(70) For every subset X of $\mathbb{I}$ and for all points a, b of $\mathbb{I}$ such that $a \leq b$ and $X =]a,b[$ holds X is open.

(71) For every subset X of $I(01)$ and for all points a, b of $\mathbb{I}$ such that $a \leq b$ and $X =]a,b[$ holds X is open.

(72) For every non empty subset X of $I(01)$ and for every point a of $\mathbb{I}$ such that $0 < a$ and $X =]0,a]$ holds X is closed.

(73) For every non empty subset X of $I(01)$ and for every point a of $\mathbb{I}$ such that $X = [a,1[$ holds X is closed.

(74) Let A be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and a, b be points of $\mathbb{I}$. Suppose A is an arc from p to q and $a < b$ and $b \neq 1$. Then there exists a non empty subset E of $I(01)$ and there exists a map f from $I(01)\restriction E$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction(A \setminus \{p\})$ such that $E =]a,b]$ and f is a homeomorphism and $f(b) = q$.

(75) Let A be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and a, b be points of $\mathbb{I}$. Suppose A is an arc from p to q and $a < b$ and $a \neq 0$. Then there exists a non empty subset E of $I(01)$ and there exists a map f from $I(01)\restriction E$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction(A \setminus \{q\})$ such that $E = [a,b[$ and f is a homeomorphism and $f(a) = p$.

(76) Let A, B be non empty subsets of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose A is an arc from p to q and B is an arc from q to p and $A \cap B = \{p,q\}$ and $p \neq q$. Then $I(01)$ and $(\mathcal{E}^2_{\rm T})\restriction((A \setminus \{p\}) \cup (B \setminus \{p\}))$ are homeomorphic.

(77) For every simple closed curve D and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in D$ holds $(\mathcal{E}^2_{\rm T})\restriction(D \setminus \{p\})$ and $I(01)$ are homeomorphic.

(78) Let D be a simple closed curve and p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in D$ and $q \in D$, then $(\mathcal{E}^2_{\rm T})\restriction(D \setminus \{p\})$ and $(\mathcal{E}^2_{\rm T})\restriction(D \setminus \{q\})$ are homeomorphic.

(79) Let C be a non empty subset of $\mathcal{E}^2_{\rm T}$ and E be a subset of $I(01)$. Suppose there exist points p_1, p_2 of $\mathbb{I}$ such that $p_1 < p_2$ and $E = [p_1, p_2]$ and $I(01)\restriction E$ and $(\mathcal{E}^2_{\rm T})\restriction C$ are homeomorphic. Then there exist points s_1, s_2 of $\mathcal{E}^2_{\rm T}$ such that C is an arc from s_1 to s_2.

(80) Let D_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$, f be a map from $(\mathcal{E}^2_{\rm T})\restriction D_1$ into $I(01)$, and C be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose f is a homeomorphism and $C \subseteq D_1$ and there exist points p_1, p_2 of $\mathbb{I}$ such that $p_1 < p_2$ and $f^\circ C = [p_1, p_2]$. Then there exist points s_1, s_2 of $\mathcal{E}^2_{\rm T}$ such that C is an arc from s_1 to s_2.

(81) Let D be a simple closed curve and C be a non empty compact connected subset of $\mathcal{E}^2_{\rm T}$. Suppose $C \subseteq D$. Then $C = D$ or there exist points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ such that C is an arc from p_1 to p_2 or there exists a point p of $\mathcal{E}^2_{\rm T}$ such that $C = \{p\}$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Józef Białas. Group and field definitions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/realset1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[5] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[6] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_4.html.

[7] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Journal of Formalized Mathematics*, 9, 1997. http://mizar.org/JFM/Vol9/pscomp_1.html.

[8] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/compts_1.html.

[9] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/tops_2.html.

[10] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[11] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces — fundamental concepts. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topmetr.html.

[12] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\rm T}$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[13] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\rm T}$. Simple closed curves. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal2.html.

[14] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Journal of Formalized Mathematics*, 11, 1999. http://mizar.org/JFM/Vol11/integra1.html.

[15] Adam Grabowski. Properties of the product of compact topological spaces. *Journal of Formalized Mathematics*, 11, 1999. http://mizar.org/JFM/Vol11/borsuk_3.html.

[16] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/tsep_1.html.

[17] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_4.html.

[18] Yatsuka Nakamura. Half open intervals in real numbers. *Journal of Formalized Mathematics*, 14, 2002. http://mizar.org/JFM/Vol14/rcomp_2.html.

[19] Yatsuka Nakamura and Jarosław Kotowicz. The Jordan's property for certain subsets of the plane. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/jordan1.html.

[20] Beata Padlewska. Connected spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/connsp_1.html.

[21] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[22] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rcomp_1.html.

[23] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[24] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/borsuk_1.html.

[25] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[26] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[27] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[28] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/t_0topsp.html.

Received August 7, 2002

Published January 2, 2004