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On Defining Functions on Binary Trees[1]

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Summary. This article is a continuation of an article on defining functions on trees (see [6]). In this article we develop terminology specialized for binary trees, first defining binary trees and binary grammars. We recast the induction principle for the set of parse trees of binary grammars and the scheme of defining functions inductively with the set as domain. We conclude with defining the scheme of defining such functions by lambda-like expressions.

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The articles [12], [9], [15], [14], [16], [17], [13], [7], [8], [5], [11], [10], [1], [2], [3], [4], and [6] provide the notation and terminology for this paper.

Let D be a non empty set and let t be a tree decorated with elements of D. The root label of t is an element of D and is defined by:

(Def. 1) The root label of $t = t(\emptyset)$.

Next we state two propositions:

(1) Let D be a non empty set and t be a tree decorated with elements of D. Then the roots of $\langle t \rangle = \langle$the root label of $t\rangle$.

(2) Let D be a non empty set and t_1, t_2 be trees decorated with elements of D. Then the roots of $\langle t_1, t_2 \rangle = \langle$the root label of t_1, the root label of $t_2\rangle$.

Let I_1 be a tree. We say that I_1 is binary if and only if:

(Def. 2) For every element t of I_1 such that $t \notin \mathrm{Leaves}(I_1)$ holds $\mathrm{succ}\, t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$.

The following propositions are true:

(3) For every tree T and for every element t of T holds $t \in \mathrm{Leaves}(T)$ iff $t \frown \langle 0 \rangle \notin T$.

(4) For every tree T and for every element t of T holds $t \in \mathrm{Leaves}(T)$ iff it is not true that there exists a natural number n such that $t \frown \langle n \rangle \in T$.

(5) For every tree T and for every element t of T holds $\mathrm{succ}\, t = \emptyset$ iff $t \in \mathrm{Leaves}(T)$.

(6) The elementary tree of 0 is binary.

(7) The elementary tree of 2 is binary.

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Let us observe that there exists a tree which is binary and finite.

Let I_1 be a decorated tree. We say that I_1 is binary if and only if:

(Def. 3) $\operatorname{dom} I_1$ is binary.

Let D be a non empty set. Observe that there exists a tree decorated with elements of D which is binary and finite.

Let us mention that there exists a decorated tree which is binary and finite.

One can check that every tree which is binary is also finite-order.

The following four propositions are true:

(8) Let T_0, T_1 be trees and t be an element of $\widehat{T_0, T_1}$. Then

(i) for every element p of T_0 such that $t = \langle 0 \rangle \frown p$ holds $t \in \mathrm{Leaves}(\widehat{T_0, T_1})$ iff $p \in \mathrm{Leaves}(T_0)$, and

(ii) for every element p of T_1 such that $t = \langle 1 \rangle \frown p$ holds $t \in \mathrm{Leaves}(\widehat{T_0, T_1})$ iff $p \in \mathrm{Leaves}(T_1)$.

(9) Let T_0, T_1 be trees and t be an element of $\widehat{T_0, T_1}$. Then

(i) if $t = \emptyset$, then $\operatorname{succ} t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$,

(ii) for every element p of T_0 such that $t = \langle 0 \rangle \frown p$ and for every finite sequence s_1 holds $s_1 \in \operatorname{succ} p$ iff $\langle 0 \rangle \frown s_1 \in \operatorname{succ} t$, and

(iii) for every element p of T_1 such that $t = \langle 1 \rangle \frown p$ and for every finite sequence s_1 holds $s_1 \in \operatorname{succ} p$ iff $\langle 1 \rangle \frown s_1 \in \operatorname{succ} t$.

(10) For all trees T_1, T_2 holds T_1 is binary and T_2 is binary iff $\widehat{T_1, T_2}$ is binary.

(11) For all decorated trees T_1, T_2 and for every set x holds T_1 is binary and T_2 is binary iff x-tree(T_1, T_2) is binary.

Let D be a non empty set, let x be an element of D, and let T_1, T_2 be binary finite trees decorated with elements of D. Then x-tree(T_1, T_2) is a binary finite tree decorated with elements of D.

Let I_1 be a non empty tree construction structure. We say that I_1 is binary if and only if:

(Def. 4) For every symbol s of I_1 and for every finite sequence p such that $s \Rightarrow p$ there exist symbols x_1, x_2 of I_1 such that $p = \langle x_1, x_2 \rangle$.

One can check that there exists a non empty tree construction structure which is binary and strict and has terminals, nonterminals, and useful nonterminals.

The scheme *BinDTConstrStrEx* deals with a non empty set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

> There exists a binary strict non empty tree construction structure G such that the carrier of $G = \mathcal{A}$ and for all symbols x, y, z of G holds $x \Rightarrow \langle y, z \rangle$ iff $\mathcal{P}[x, y, z]$

for all values of the parameters.

The following proposition is true

(12) Let G be a binary non empty tree construction structure with terminals and nonterminals, t_3 be a finite sequence of elements of $\mathrm{TS}(G)$, and n_1 be a symbol of G. Suppose $n_1 \Rightarrow$ the roots of t_3. Then

(i) n_1 is a nonterminal of G,

(ii) $\operatorname{dom} t_3 = \{1, 2\}$,

(iii) $1 \in \operatorname{dom} t_3$,

(iv) $2 \in \operatorname{dom} t_3$, and

(v) there exist elements t_4, t_5 of $\mathrm{TS}(G)$ such that the roots of $t_3 = \langle$the root label of t_4, the root label of $t_5\rangle$ and $t_4 = t_3(1)$ and $t_5 = t_3(2)$ and n_1-tree$(t_3) = n_1$-tree(t_4, t_5) and $t_4 \in \operatorname{rng} t_3$ and $t_5 \in \operatorname{rng} t_3$.

Now we present three schemes. The scheme *BinDTConstrInd* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals and nonterminals and a unary predicate $\mathcal{P}$, and states that:

For every element t of $\mathrm{TS}(\mathcal{A})$ holds $\mathcal{P}[t]$

provided the parameters meet the following requirements:

- For every terminal s of $\mathcal{A}$ holds $\mathcal{P}$[the root tree of s], and
- Let n_1 be a nonterminal of $\mathcal{A}$ and t_4, t_5 be elements of $\mathrm{TS}(\mathcal{A})$. Suppose $n_1 \Rightarrow \langle$the root label of t_4, the root label of $t_5\rangle$ and $\mathcal{P}[t_4]$ and $\mathcal{P}[t_5]$. Then $\mathcal{P}[n_1\text{-tree}(t_4,t_5)]$.

The scheme *BinDTConstrIndDef* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a 5-ary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

There exists a function f from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$ such that

(i) for every terminal t of $\mathcal{A}$ holds $f(\text{the root tree of } t) = \mathcal{F}(t)$, and

(ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_4, t_5 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_4 and $r_2 =$ the root label of t_5 and $n_1 \Rightarrow \langle r_1, r_2\rangle$ and for all elements x_3, x_4 of $\mathcal{B}$ such that $x_3 = f(t_4)$ and $x_4 = f(t_5)$ holds $f(n_1\text{-tree}(t_4,t_5)) = \mathcal{G}(n_1,r_1,r_2,x_3,x_4)$

for all values of the parameters.

The scheme *BinDTConstrUniqDef* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, a non empty set $\mathcal{B}$, functions $\mathcal{C}$, $\mathcal{D}$ from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a 5-ary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- (i) For every terminal t of $\mathcal{A}$ holds $\mathcal{C}(\text{the root tree of } t) = \mathcal{F}(t)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_4, t_5 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_4 and $r_2 =$ the root label of t_5 and $n_1 \Rightarrow \langle r_1, r_2\rangle$ and for all elements x_3, x_4 of $\mathcal{B}$ such that $x_3 = \mathcal{C}(t_4)$ and $x_4 = \mathcal{C}(t_5)$ holds $\mathcal{C}(n_1\text{-tree}(t_4,t_5)) = \mathcal{G}(n_1,r_1,r_2,x_3,x_4)$,
  and
- (i) For every terminal t of $\mathcal{A}$ holds $\mathcal{D}(\text{the root tree of } t) = \mathcal{F}(t)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_4, t_5 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_4 and $r_2 =$ the root label of t_5 and $n_1 \Rightarrow \langle r_1, r_2\rangle$ and for all elements x_3, x_4 of $\mathcal{B}$ such that $x_3 = \mathcal{D}(t_4)$ and $x_4 = \mathcal{D}(t_5)$ holds $\mathcal{D}(n_1\text{-tree}(t_4,t_5)) = \mathcal{G}(n_1,r_1,r_2,x_3,x_4)$.

Let A, B, C be non empty sets, let a be an element of A, let b be an element of B, and let c be an element of C. Then $\langle a, b, c\rangle$ is an element of $[:A, B, C:]$.

Now we present two schemes. The scheme *BinDTC DefLambda* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, non empty sets $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a 4-ary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

There exists a function f from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{C}^{\mathcal{B}}$ such that

(i) for every terminal t of $\mathcal{A}$ there exists a function g from $\mathcal{B}$ into $\mathcal{C}$ such that $g = f(\text{the root tree of } t)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{F}(t,a)$, and

(ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_1, t_2 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_1 and $r_2 =$ the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2\rangle$ there exist functions g, f_1, f_2 from $\mathcal{B}$ into $\mathcal{C}$ such that $g = f(n_1\text{-tree}(t_1,t_2))$ and $f_1 = f(t_1)$ and $f_2 = f(t_2)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{G}(n_1, f_1, f_2, a)$

for all values of the parameters.

The scheme *BinDTC DefLambdaUniq* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, non empty sets $\mathcal{B}$, $\mathcal{C}$, functions $\mathcal{D}$, $\mathcal{E}$ from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{C}^{\mathcal{B}}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a 4-ary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

$\mathcal{D} = \mathcal{E}$

provided the following conditions are satisfied:

- (i) For every terminal t of $\mathcal{A}$ there exists a function g from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{D}(\text{the root tree of } t)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{F}(t, a)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_1, t_2 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_1 and $r_2 =$ the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ there exist functions g, f_1, f_2 from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{D}(n_1\text{-tree}(t_1, t_2))$ and $f_1 = \mathcal{D}(t_1)$ and $f_2 = \mathcal{D}(t_2)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{G}(n_1, f_1, f_2, a)$,
  and
- (i) For every terminal t of $\mathcal{A}$ there exists a function g from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{E}(\text{the root tree of } t)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{F}(t, a)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_1, t_2 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_1 and $r_2 =$ the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ there exist functions g, f_1, f_2 from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{E}(n_1\text{-tree}(t_1, t_2))$ and $f_1 = \mathcal{E}(t_1)$ and $f_2 = \mathcal{E}(t_2)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{G}(n_1, f_1, f_2, a)$.

Let G be a binary non empty tree construction structure with terminals and nonterminals. Note that every element of $\mathrm{TS}(G)$ is binary.

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