JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
Inst. of Computer Science, Univ. of Białystok

A Representation of Integers by Binary Arithmetics and Addition of Integers

Hisayoshi Kunimune
Shinshu University
Nagano

Yatsuka Nakamura
Shinshu University
Nagano

Summary. In this article, we introduce the new concept of 2's complement representation. Natural numbers that are congruent mod n can be represented by the same n bits binary. Using the concept introduced here, negative numbers that are congruent mod n also can be represented by the same n bit binary. We also show some properties of addition of integers using this concept.

MML Identifier: BINARI_4.

WWW: http://mizar.org/JFM/Vol15/binari_4.html

The articles [16], [21], [2], [5], [12], [11], [10], [9], [17], [13], [15], [6], [7], [1], [14], [18], [3], [20], [8], [4], and [19] provide the notation and terminology for this paper.

1. PRELIMINARIES

We use the following convention: n denotes a non empty natural number, j, k, l, m denote natural numbers, and g, h, i denote integers.

We now state a number of propositions:

(1) If $m > 0$, then $m \cdot 2 \geq m + 1$.

(2) For every natural number m holds $2^m \geq m$.

(3) For every natural number m holds $\langle\underbrace{0,\ldots,0}_{m}\rangle + \langle\underbrace{0,\ldots,0}_{m}\rangle = \langle\underbrace{0,\ldots,0}_{m}\rangle$.

(4) For every natural number k such that $k \leq l$ and $l \leq m$ holds $k = l$ or $k + 1 \leq l$ and $l \leq m$.

(5) For every non empty natural number n and for all n-tuples x, y of *Boolean* such that $x = \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $y = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds $\mathrm{carry}(x, y) = \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(6) For every non empty natural number n and for all n-tuples x, y of *Boolean* such that $x = \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $y = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds $x + y = \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(7) For every non empty natural number n and for every n-tuple F of *Boolean* such that $F = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds $\mathrm{Intval}(F) = 0$.

(8) If $l + m \leq k - 1$, then $l < k$ and $m < k$.

(9) If $g \leq h+i$ and $h < 0$ and $i < 0$, then $g < h$ and $g < i$.

(10) If $l+m \leq 2^n - 1$, then $\mathrm{add\_ovfl}(n\text{-BinarySequence}(l), n\text{-BinarySequence}(m)) = \mathit{false}$.

(11) For every non empty natural number n and for all natural numbers l, m such that $l+m \leq 2^n - 1$ holds $\mathrm{Absval}((n\text{-BinarySequence}(l)) + (n\text{-BinarySequence}(m))) = l+m$.

(12) For every non empty natural number n and for every n-tuple z of *Boolean* such that $z_n = \mathit{true}$ holds $\mathrm{Absval}(z) \geq 2^{n -' 1}$.

(13) If $l+m \leq 2^{n -' 1} - 1$, then $(\mathrm{carry}(n\text{-BinarySequence}(l), n\text{-BinarySequence}(m)))_n = \mathit{false}$.

(14) For every non empty natural number n such that $l+m \leq 2^{n -' 1} - 1$ holds $\mathrm{Intval}((n\text{-BinarySequence}(l)) + (n\text{-BinarySequence}(m))) = l+m$.

(15) For every 1-tuple z of *Boolean* such that $z = \langle \mathit{true} \rangle$ holds $\mathrm{Intval}(z) = -1$.

(16) For every 1-tuple z of *Boolean* such that $z = \langle \mathit{false} \rangle$ holds $\mathrm{Intval}(z) = 0$.

(17) For every boolean set x holds $\mathit{true} \vee x = \mathit{true}$.

(18) For every non empty natural number n holds $0 \leq 2^{n -' 1} - 1$ and $-2^{n -' 1} \leq 0$.

(19) For all n-tuples x, y of *Boolean* such that $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ and $y = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ holds x and y are summable.

(20) $i \cdot n \bmod n = 0$.

2. Majorant Power

Let m, j be natural numbers. The functor $\mathrm{MajP}(m, j)$ yields a natural number and is defined by:

(Def. 1) $2^{\mathrm{MajP}(m,j)} \geq j$ and $\mathrm{MajP}(m, j) \geq m$ and for every natural number k such that $2^k \geq j$ and $k \geq m$ holds $k \geq \mathrm{MajP}(m, j)$.

We now state several propositions:

(21) If $j \geq k$, then $\mathrm{MajP}(m, j) \geq \mathrm{MajP}(m, k)$.

(22) If $l \geq m$, then $\mathrm{MajP}(l, j) \geq \mathrm{MajP}(m, j)$.

(23) If $m \geq 1$, then $\mathrm{MajP}(m, 1) = m$.

(24) If $j \leq 2^m$, then $\mathrm{MajP}(m, j) = m$.

(25) If $j > 2^m$, then $\mathrm{MajP}(m, j) > m$.

3. 2's Complement

Let m be a natural number and let i be an integer. The functor $\mathrm{2sComplement}(m,i)$ yields a m-tuple of *Boolean* and is defined as follows:

(Def. 2) $\mathrm{2sComplement}(m,i) = \begin{cases} m\text{-BinarySequence}(|2^{\mathrm{MajP}(m,|i|)} + i|), & \text{if } i < 0, \\ m\text{-BinarySequence}(|i|), & \text{otherwise.} \end{cases}$

The following propositions are true:

(26) For every natural number m holds $\mathrm{2sComplement}(m, 0) = \langle \underbrace{0, \ldots, 0}_{m} \rangle$.

(27) For every integer i such that $i \leq 2^{n -' 1} - 1$ and $-2^{n -' 1} \leq i$ holds $\mathrm{Intval}(\mathrm{2sComplement}(n, i)) = i$.

(28) For all integers h, i such that $h \geq 0$ and $i \geq 0$ or $h < 0$ and $i < 0$ but $h \bmod 2^n = i \bmod 2^n$ holds $\mathrm{2sComplement}(n,h) = \mathrm{2sComplement}(n,i)$.

(29) For all integers h, i such that $h \geq 0$ and $i \geq 0$ or $h < 0$ and $i < 0$ but $h \equiv i (\bmod 2^n)$ holds $\mathrm{2sComplement}(n,h) = \mathrm{2sComplement}(n,i)$.

(30) For all natural numbers l, m such that $l \bmod 2^n = m \bmod 2^n$ holds n-BinarySequence$(l) =$ n-BinarySequence(m).

(31) For all natural numbers l, m such that $l \equiv m (\bmod 2^n)$ holds n-BinarySequence$(l) =$ n-BinarySequence(m).

(32) For every natural number j such that $1 \leq j$ and $j \leq n$ holds $(\mathrm{2sComplement}(n+1,i))_j = (\mathrm{2sComplement}(n,i))_j$.

(33) There exists an element x of *Boolean* such that $\mathrm{2sComplement}(m+1,i) = (\mathrm{2sComplement}(m,i)) \frown \langle x \rangle$.

(34) There exists an element x of *Boolean* such that $(m+1)$-BinarySequence$(l) = (m$-BinarySequence$(l)) \frown \langle x \rangle$.

(35) Let n be a non empty natural number. Suppose $-2^n \leq h+i$ and $h < 0$ and $i < 0$ and $-2^{n -' 1} \leq h$ and $-2^{n -' 1} \leq i$. Then $(\mathrm{carry}(\mathrm{2sComplement}(n+1,h), \mathrm{2sComplement}(n+1,i)))_{n+1} = \mathit{true}$.

(36) For every non empty natural number n such that $-2^{n -' 1} \leq h+i$ and $h+i \leq 2^{n -' 1} - 1$ and $h \geq 0$ and $i \geq 0$ holds $\mathrm{Intval}(\mathrm{2sComplement}(n,h) + \mathrm{2sComplement}(n,i)) = h+i$.

(37) Let n be a non empty natural number. Suppose $-2^{(n+1) -' 1} \leq h+i$ and $h+i \leq 2^{(n+1) -' 1} - 1$ and $h < 0$ and $i < 0$ and $-2^{n -' 1} \leq h$ and $-2^{n -' 1} \leq i$. Then $\mathrm{Intval}(\mathrm{2sComplement}(n+1,h) + \mathrm{2sComplement}(n+1,i)) = h+i$.

(38) Let n be a non empty natural number. Suppose that $-2^{n -' 1} \leq h$ and $h \leq 2^{n -' 1} - 1$ and $-2^{n -' 1} \leq i$ and $i \leq 2^{n -' 1} - 1$ and $-2^{n -' 1} \leq h+i$ and $h+i \leq 2^{n -' 1} - 1$ and $h \geq 0$ and $i < 0$ or $h < 0$ and $i \geq 0$. Then $\mathrm{Intval}(\mathrm{2sComplement}(n,h) + \mathrm{2sComplement}(n,i)) = h+i$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[5] Czesław Byliński. A classical first order language. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/cqc_lang.html`.

[6] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/euclid.html`.

[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[8] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/vectsp_1.html`.

[9] Robert Milewski. Binary arithmetics. Binary sequences. *Journal of Formalized Mathematics*, 10, 1998. `http://mizar.org/JFM/Vol10/binari_3.html`.

[10] Yasuho Mizuhara and Takaya Nishiyama. Binary arithmetics, addition and subtraction of integers. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/binari_2.html`.

[11] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/binarith.html`.

[12] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/power.html.

[13] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/series_1.html.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[15] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[16] Michał J. Trybulec. Integers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/int_1.html.

[17] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/group_1.html.

[18] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[19] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[20] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[21] Edmund Woronowicz. Many-argument relations. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/margrel1.html.

Received January 30, 2003

Published January 2, 2004