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On Some Properties of Real Hilbert Space. Part I

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Summary. In this paper, we first introduce the notion of summability of an infinite set of vectors of real Hilbert space, without using index sets. Further we introduce the notion of weak summability, which is weaker than that of summability. Then, several statements for summable sets and weakly summable ones are proved. In the last part of the paper, we give a necessary and sufficient condition for summability of an infinite set of vectors of real Hilbert space as our main theorem. The last theorem is due to [8].

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The articles [17], [21], [6], [1], [18], [9], [22], [4], [5], [7], [12], [19], [14], [15], [13], [3], [10], [16], [11], [2], [20], and [23] provide the notation and terminology for this paper.

1. PRELIMINARIES

In this paper X is a real unitary space, x is a point of X, and i is a natural number.

Let us consider X. Let us assume that the addition of X is commutative and associative and the addition of X has a unity. Let Y be a finite subset of X. The functor $\mathrm{Setsum}(Y)$ yields an element of X and is defined by the condition (Def. 1).

(Def. 1) There exists a finite sequence p of elements of the carrier of X such that p is one-to-one and $\mathrm{rng}\, p = Y$ and $\mathrm{Setsum}(Y) =$ the addition of $X \odot p$.

The following two propositions are true:

(1) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y be a finite subset of X and I be a function from the carrier of X into the carrier of X. Suppose $Y \subseteq \mathrm{dom}\, I$ and for every set x such that $x \in \mathrm{dom}\, I$ holds $I(x) = x$. Then $\mathrm{Setsum}(Y) = \mathrm{setopfunc}(Y,$ the carrier of X, the carrier of X, $I,$ the addition of $X)$.

(2) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y_1, Y_2 be finite subsets of X. Suppose Y_1 misses Y_2. Let Z be a finite subset of X. If $Z = Y_1 \cup Y_2$, then $\mathrm{Setsum}(Z) = \mathrm{Setsum}(Y_1) + \mathrm{Setsum}(Y_2)$.

2. Summability

Let us consider X and let Y be a subset of X. We say that Y is summable_set if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists x such that for every real number e if $e > 0$, then there exists a finite subset Y_0 of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $\|x - \mathrm{Setsum}(Y_1)\| < e$.

Let us consider X and let Y be a subset of X. Let us assume that Y is summable_set. The functor $\mathrm{sum}\, Y$ yielding a point of X is defined by the condition (Def. 3).

(Def. 3) Let e be a real number. Suppose $e > 0$. Then there exists a finite subset Y_0 of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $\|\mathrm{sum}\, Y - \mathrm{Setsum}(Y_1)\| < e$.

Let us consider X and let L be a linear functional in X. We say that L is Bounded if and only if:

(Def. 4) There exists a real number K such that $K > 0$ and for every x holds $|L(x)| \leq K \cdot \|x\|$.

Let us consider X and let Y be a subset of X. We say that Y is weakly summable_set if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists x such that for every linear functional L in X if L is Bounded, then for every real number e such that $e > 0$ there exists a finite subset Y_0 of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $|L(x - \mathrm{Setsum}(Y_1))| < e$.

Let us consider X, let Y be a subset of X, and let L be a functional in X. We say that Y is summable set by L if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a real number r such that for every real number e if $e > 0$, then there exists a finite subset Y_0 of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $|r - \mathrm{setopfunc}(Y_1, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})| < e$.

Let us consider X, let Y be a subset of X, and let L be a functional in X. Let us assume that Y is summable set by L. The functor $\mathrm{SumByfunc}(Y, L)$ yields a real number and is defined by the condition (Def. 7).

(Def. 7) Let e be a real number. Suppose $e > 0$. Then there exists a finite subset Y_0 of X such that

(i) Y_0 is non empty,

(ii) $Y_0 \subseteq Y$, and

(iii) for every finite subset Y_1 of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $|\mathrm{SumByfunc}(Y, L) - \mathrm{setopfunc}(Y_1, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})| < e$.

One can prove the following propositions:

(3) For every subset Y of X such that Y is summable_set holds Y is weakly summable_set.

(4) Let L be a linear functional in X and p be a finite sequence of elements of the carrier of X. Suppose $\mathrm{len}\, p \geq 1$. Let q be a finite sequence of elements of $\mathbb{R}$. Suppose $\mathrm{dom}\, p = \mathrm{dom}\, q$ and for every i such that $i \in \mathrm{dom}\, q$ holds $q(i) = L(p(i))$. Then $L(\text{the addition of } X \odot p) = +_{\mathbb{R}} \odot q$.

(5) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let S be a finite subset of X. Suppose S is non empty. Let L be a linear functional in X. Then $L(\mathrm{Setsum}(S)) = \mathrm{setopfunc}(S, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})$.

(6) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y be a subset of X. Suppose Y is weakly summable_set. Then there exists x such that for every linear functional L in X if L is Bounded, then for every real number e such that $e > 0$ there exists a finite subset Y_0 of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $|L(x) - \mathrm{setopfunc}(Y_1, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})| < e$.

(7) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y be a subset of X. Suppose Y is weakly summable_set. Let L be a linear functional in X. If L is Bounded, then Y is summable set by L.

(8) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y be a subset of X. Suppose Y is summable_set. Let L be a linear functional in X. If L is Bounded, then Y is summable set by L.

(9) For every finite subset Y of X such that Y is non empty holds Y is summable_set.

3. Necessary and Sufficient Condition for Summability

One can prove the following proposition

(10) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity and X is a Hilbert space. Let Y be a subset of X. Then Y is summable_set if and only if for every real number e such that $e > 0$ there exists a finite subset Y_0 of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of X such that Y_1 is non empty and $Y_1 \subseteq Y$ and Y_0 misses Y_1 holds $\|\mathrm{Setsum}(Y_1)\| < e$.

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