JOURNAL OF FORMALIZED MATHEMATICS
Volume 14, Released 2002, Published 2003
Inst. of Computer Science, Univ. of Białystok

On Ordering of Bags

Gilbert Lee
University of Alberta
Edmonton

Piotr Rudnicki
University of Alberta
Edmonton

Summary. We present a Mizar formalization of chapter 4.4 of [8] devoted to special orderings in additive monoids to be used for ordering terms in multivariate polynomials. We have extended the treatment to the case of infinite number of variables. It turns out that in such case admissible orderings are not necessarily well orderings.

MML Identifier: BAGORDER.

WWW: http://mizar.org/JFM/Vol14/bagorder.html

The articles [33], [12], [41], [42], [44], [43], [16], [37], [3], [34], [4], [38], [40], [29], [36], [19], [2], [9], [21], [6], [1], [28], [35], [24], [5], [23], [27], [7], [20], [30], [15], [26], [18], [17], [11], [10], [14], [25], [31], [39], [32], [13], and [22] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all sets x, y, z such that $z \in x$ and $z \in y$ holds $x \setminus \{z\} = y \setminus \{z\}$ iff $x = y$.

(2) For all natural numbers n, k holds $k \in \mathrm{Seg}\, n$ iff $k - 1$ is a natural number and $k - 1 < n$.

Let f be a natural-yielding function and let X be a set. Observe that $f \restriction X$ is natural-yielding.

Let f be a finite-support function and let X be a set. One can check that $f \restriction X$ is finite-support.

One can prove the following three propositions:

(3) For every function f and for every set x such that $x \in \mathrm{dom}\, f$ holds $f \cdot \langle x \rangle = \langle f(x) \rangle$.

(4) Let f, g, h be functions. Suppose $\mathrm{dom}\, f = \mathrm{dom}\, g$ and $\mathrm{rng}\, f \subseteq \mathrm{dom}\, h$ and $\mathrm{rng}\, g \subseteq \mathrm{dom}\, h$ and f and g are fiberwise equipotent. Then $h \cdot f$ and $h \cdot g$ are fiberwise equipotent.

(5) For every finite sequence f_1 of elements of $\mathbb{N}$ holds $\sum f_1 = 0$ iff $f_1 = \mathrm{len}\, f_1 \mapsto 0$.

Let n, i, j be natural numbers and let b be a many sorted set indexed by n. The functor $\langle b(i), \ldots, b(j) \rangle$ yielding a many sorted set indexed by $j -' i$ is defined as follows:

(Def. 1) For every natural number k such that $k \in j -' i$ holds $\langle b(i), \ldots, b(j) \rangle(k) = b(i + k)$.

Let n, i, j be natural numbers and let b be a natural-yielding many sorted set indexed by n. Note that $\langle b(i), \ldots, b(j) \rangle$ is natural-yielding.

Let n, i, j be natural numbers and let b be a finite-support many sorted set indexed by n. One can verify that $\langle b(i), \ldots, b(j) \rangle$ is finite-support.

We now state the proposition

(6) Let n, i be natural numbers and a, b be many sorted sets indexed by n. Then $a = b$ if and only if the following conditions are satisfied:

(i) $\langle a(0), \ldots, a(i+1) \rangle = \langle b(0), \ldots, b(i+1) \rangle$, and

(ii) $\langle a(i+1), \ldots, a(n) \rangle = \langle b(i+1), \ldots, b(n) \rangle$.

Let x be a non empty set and let n be a non empty natural number. The functor $\mathrm{Fin}(x,n)$ is defined by:

(Def. 2) $\mathrm{Fin}(x,n) = \{y; y$ ranges over elements of 2^x: y is finite $\wedge$ $y$is non empty $\wedge \overline{\overline{y}} \leq n\}$.

Let x be a non empty set and let n be a non empty natural number. Observe that $\mathrm{Fin}(x,n)$ is non empty.

One can prove the following three propositions:

(7) Let R be an antisymmetric transitive non empty relational structure and X be a finite subset of R. Suppose $X \neq \emptyset$. Then there exists an element x of R such that $x \in X$ and x is maximal w.r.t. X, the internal relation of R.

(8) Let R be an antisymmetric transitive non empty relational structure and X be a finite subset of R. Suppose $X \neq \emptyset$. Then there exists an element x of R such that $x \in X$ and x is minimal w.r.t. X, the internal relation of R.

(9) Let R be a non empty antisymmetric transitive relational structure and f be a sequence of R. Suppose f is descending. Let j, i be natural numbers. If $i < j$, then $f(i) \neq f(j)$ and $\langle f(j), f(i) \rangle \in$ the internal relation of R.

Let R be a non empty relational structure and let s be a sequence of R. We say that s is non-increasing if and only if:

(Def. 3) For every natural number i holds $\langle s(i+1), s(i) \rangle \in$ the internal relation of R.

The following three propositions are true:

(10) Let R be a non empty transitive relational structure and f be a sequence of R. Suppose f is non-increasing. Let j, i be natural numbers. If $i < j$, then $\langle f(j), f(i) \rangle \in$ the internal relation of R.

(11) Let R be a non empty transitive relational structure and s be a sequence of R. Suppose R is well founded and s is non-increasing. Then there exists a natural number p such that for every natural number r if $p \leq r$, then $s(p) = s(r)$.

(12) Let X be a set, a be an element of X, A be a finite subset of X, and R be an order in X. If $A = \{a\}$ and R linearly orders A, then $\mathrm{SgmX}(R,A) = \langle a \rangle$.

2. More About Bags

Let n be an ordinal number and let b be a bag of n. The functor $\mathrm{TotDegree}\, b$ yields a natural number and is defined by:

(Def. 4) There exists a finite sequence f of elements of $\mathbb{N}$ such that $\mathrm{TotDegree}\, b = \sum f$ and $f = b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\, b)$.

One can prove the following propositions:

(13) Let n be an ordinal number, b be a bag of n, s be a finite subset of n, and f, g be finite sequences of elements of $\mathbb{N}$. If $f = b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\, b)$ and $g = b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\, b \cup s)$, then $\sum f = \sum g$.

(14) For every ordinal number n and for all bags a, b of n holds $\mathrm{TotDegree}(a+b) = \mathrm{TotDegree}\, a + \mathrm{TotDegree}\, b$.

(15) For every ordinal number n and for all bags a, b of n such that $b \mid a$ holds $\mathrm{TotDegree}(a -' b) = \mathrm{TotDegree}\,a - \mathrm{TotDegree}\,b$.

(16) For every ordinal number n and for every bag b of n holds $\mathrm{TotDegree}\,b = 0$ iff $b = \mathrm{EmptyBag}\,n$.

(17) For all natural numbers i, j, n holds $\langle(\mathrm{EmptyBag}\,n)(i), \ldots, (\mathrm{EmptyBag}\,n)(j)\rangle = \mathrm{EmptyBag}(j -' i)$.

(18) For all natural numbers i, j, n and for all bags a, b of n holds $\langle(a+b)(i), \ldots, (a+b)(j)\rangle = \langle a(i), \ldots, a(j)\rangle + \langle b(i), \ldots, b(j)\rangle$.

(19) For every set X holds $\mathrm{support}\,\mathrm{EmptyBag}\,X = \emptyset$.

(20) For every set X and for every bag b of X such that $\mathrm{support}\,b = \emptyset$ holds $b = \mathrm{EmptyBag}\,X$.

(21) For all ordinal numbers n, m and for every bag b of n such that $m \in n$ holds $b \restriction m$ is a bag of m.

(22) For every ordinal number n and for all bags a, b of n such that $b \mid a$ holds $\mathrm{support}\,b \subseteq \mathrm{support}\,a$.

3. Some Special Orders

Let n be a set. A term order of n is an order in $\mathrm{Bags}\,n$.

Let n be an ordinal number. We introduce $\mathrm{LexOrder}\,n$ as a synonym of $\mathrm{BagOrder}\,n$.

Let n be an ordinal number and let T be a term order of n. We say that T is admissible if and only if the conditions (Def. 7) are satisfied.

(Def. 7)[1] (i) T is strongly connected in $\mathrm{Bags}\,n$,

(ii) for every bag a of n holds $\langle \mathrm{EmptyBag}\,n, a\rangle \in T$, and

(iii) for all bags a, b, c of n such that $\langle a, b\rangle \in T$ holds $\langle a+c, b+c\rangle \in T$.

We now state the proposition

(23) For every ordinal number n holds $\mathrm{LexOrder}\,n$ is admissible.

Let n be an ordinal number. One can verify that there exists a term order of n which is admissible.

Let n be an ordinal number. Observe that $\mathrm{LexOrder}\,n$ is admissible.

We now state the proposition

(24) For every infinite ordinal number o holds $\mathrm{LexOrder}\,o$ is non well-ordering.

Let n be an ordinal number. The functor $\mathrm{InvLexOrder}\,n$ yielding a term order of n is defined by the condition (Def. 8).

(Def. 8) Let p, q be bags of n. Then $\langle p, q\rangle \in \mathrm{InvLexOrder}\,n$ if and only if one of the following conditions is satisfied:

(i) $p = q$, or

(ii) there exists an ordinal number i such that $i \in n$ and $p(i) < q(i)$ and for every ordinal number k such that $i \in k$ and $k \in n$ holds $p(k) = q(k)$.

We now state the proposition

(25) For every ordinal number n holds $\mathrm{InvLexOrder}\,n$ is admissible.

Let n be an ordinal number. Observe that $\mathrm{InvLexOrder}\,n$ is admissible.

One can prove the following proposition

[1] The definitions (Def. 5) and (Def. 6) have been removed.

(26) For every ordinal number o holds $\mathrm{InvLexOrder}\,o$ is well-ordering.

Let n be an ordinal number and let o be a term order of n. Let us assume that for all bags a, b, c of n such that $\langle a, b\rangle \in o$ holds $\langle a+c, b+c\rangle \in o$. The functor $\mathrm{Graded}\,o$ yielding a term order of n is defined as follows:

(Def. 9) For all bags a, b of n holds $\langle a, b\rangle \in \mathrm{Graded}\,o$ iff $\mathrm{TotDegree}\,a < \mathrm{TotDegree}\,b$ or $\mathrm{TotDegree}\,a = \mathrm{TotDegree}\,b$ and $\langle a, b\rangle \in o$.

One can prove the following proposition

(27) Let n be an ordinal number and o be a term order of n. Suppose for all bags a, b, c of n such that $\langle a, b\rangle \in o$ holds $\langle a+c, b+c\rangle \in o$ and o is strongly connected in $\mathrm{Bags}\,n$. Then $\mathrm{Graded}\,o$ is admissible.

Let n be an ordinal number. The functor $\mathrm{GrLexOrder}\,n$ yields a term order of n and is defined as follows:

(Def. 10) $\mathrm{GrLexOrder}\,n = \mathrm{Graded}\,\mathrm{LexOrder}\,n$.

The functor $\mathrm{GrInvLexOrder}\,n$ yields a term order of n and is defined by:

(Def. 11) $\mathrm{GrInvLexOrder}\,n = \mathrm{Graded}\,\mathrm{InvLexOrder}\,n$.

We now state the proposition

(28) For every ordinal number n holds $\mathrm{GrLexOrder}\,n$ is admissible.

Let n be an ordinal number. Observe that $\mathrm{GrLexOrder}\,n$ is admissible.
We now state two propositions:

(29) For every infinite ordinal number o holds $\mathrm{GrLexOrder}\,o$ is non well-ordering.

(30) For every ordinal number n holds $\mathrm{GrInvLexOrder}\,n$ is admissible.

Let n be an ordinal number. One can verify that $\mathrm{GrInvLexOrder}\,n$ is admissible.
We now state the proposition

(31) For every ordinal number o holds $\mathrm{GrInvLexOrder}\,o$ is well-ordering.

Let i, n be natural numbers, let o_1 be a term order of $i+1$, and let o_2 be a term order of $n -' (i+1)$. The functor $\mathrm{BlockOrder}(i,n,o_1,o_2)$ yielding a term order of n is defined by the condition (Def. 12).

(Def. 12) Let p, q be bags of n. Then $\langle p, q\rangle \in \mathrm{BlockOrder}(i,n,o_1,o_2)$ if and only if one of the following conditions is satisfied:

(i) $\langle p(0),\ldots,p(i+1)\rangle \neq \langle q(0),\ldots,q(i+1)\rangle$ and $\langle\langle p(0),\ldots,p(i+1)\rangle, \langle q(0),\ldots,q(i+1)\rangle\rangle \in o_1$, or

(ii) $\langle p(0),\ldots,p(i+1)\rangle = \langle q(0),\ldots,q(i+1)\rangle$ and $\langle\langle p(i+1),\ldots,p(n)\rangle, \langle q(i+1),\ldots,q(n)\rangle\rangle \in o_2$.

We now state the proposition

(32) Let i, n be natural numbers, o_1 be a term order of $i+1$, and o_2 be a term order of $n -' (i+1)$. If o_1 is admissible and o_2 is admissible, then $\mathrm{BlockOrder}(i,n,o_1,o_2)$ is admissible.

Let n be a natural number. The functor $\mathrm{NaivelyOrderedBags}\,n$ yielding a strict relational structure is defined by the conditions (Def. 13).

(Def. 13)(i) The carrier of $\mathrm{NaivelyOrderedBags}\,n = \mathrm{Bags}\,n$, and

(ii) for all bags x, y of n holds $\langle x, y\rangle \in$ the internal relation of $\mathrm{NaivelyOrderedBags}\,n$ iff $x \mid y$.

Next we state three propositions:

(33) For every natural number n holds the carrier of $\prod(n \longmapsto \mathrm{OrderedNAT}) = \mathrm{Bags}\,n$.

(34) For every natural number n holds $\mathrm{NaivelyOrderedBags}\,n = \prod(n \longmapsto \mathrm{OrderedNAT})$.

(35) Let n be a natural number and o be a term order of n. Suppose o is admissible. Then the internal relation of $\mathrm{NaivelyOrderedBags}\,n \subseteq o$ and o is well-ordering.

4. Ordering of Finite Subsets

Let R be a connected non empty poset and let X be an element of Fin (the carrier of R). Let us assume that X is non empty. The functor PosetMin X yields an element of R and is defined as follows:

(Def. 14) PosetMin $X \in X$ and PosetMin X is minimal w.r.t. X, the internal relation of R.

The functor PosetMax X yields an element of R and is defined as follows:

(Def. 15) PosetMax $X \in X$ and PosetMax X is maximal w.r.t. X, the internal relation of R.

Let R be a connected non empty poset. The functor FinOrd-Approx R yielding a function from $\mathbb{N}$ into $2^{[:\mathrm{Fin}\,(\text{the carrier of } R),\, \mathrm{Fin}\,(\text{the carrier of } R):]}$ is defined by the conditions (Def. 16).

(Def. 16)(i) dom FinOrd-Approx $R = \mathbb{N}$,

(ii) $(\text{FinOrd-Approx}\,R)(0) = \{\langle x, y\rangle; x$ ranges over elements of Fin (the carrier of R), y ranges over elements of Fin (the carrier of R): $x = \emptyset \lor x \neq \emptyset \land y \neq \emptyset \land \text{PosetMax}\,x \neq \text{PosetMax}\,y \land \langle \text{PosetMax}\,x, \text{PosetMax}\,y\rangle \in$ the internal relation of $R\}$, and

(iii) for every element n of $\mathbb{N}$ holds $(\text{FinOrd-Approx}\,R)(n+1) = \{\langle x, y\rangle; x$ ranges over elements of Fin (the carrier of R), y ranges over elements of Fin (the carrier of R): $x \neq \emptyset \land y \neq \emptyset \land \text{PosetMax}\,x = \text{PosetMax}\,y \land \langle x \setminus \{\text{PosetMax}\,x\}, y \setminus \{\text{PosetMax}\,y\}\rangle \in (\text{FinOrd-Approx}\,R)(n)\}$.

We now state four propositions:

(36) Let R be a connected non empty poset and x, y be elements of Fin (the carrier of R). Then $\langle x, y\rangle \in \bigcup \text{rng FinOrd-Approx}\,R$ if and only if one of the following conditions is satisfied:

(i) $x = \emptyset$, or

(ii) $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax $x \neq$ PosetMax y and $\langle \text{PosetMax}\,x, \text{PosetMax}\,y\rangle \in$ the internal relation of R, or

(iii) $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax $x =$ PosetMax y and $\langle x \setminus \{\text{PosetMax}\,x\}, y \setminus \{\text{PosetMax}\,y\}\rangle \in \bigcup \text{rng FinOrd-Approx}\,R$.

(37) For every connected non empty poset R and for every element x of Fin (the carrier of R) such that $x \neq \emptyset$ holds $\langle x, \emptyset\rangle \notin \bigcup \text{rng FinOrd-Approx}\,R$.

(38) Let R be a connected non empty poset and a be an element of Fin (the carrier of R). Then $a \setminus \{\text{PosetMax}\,a\}$ is an element of Fin (the carrier of R).

(39) For every connected non empty poset R holds $\bigcup \text{rng FinOrd-Approx}\,R$ is an order in Fin (the carrier of R).

Let R be a connected non empty poset. The functor FinOrd R yields an order in Fin (the carrier of R) and is defined by:

(Def. 17) $\text{FinOrd}\,R = \bigcup \text{rng FinOrd-Approx}\,R$.

Let R be a connected non empty poset. The functor FinPoset R yielding a poset is defined by:

(Def. 18) $\text{FinPoset}\,R = \langle \text{Fin}\,(\text{the carrier of } R), \text{FinOrd}\,R\rangle$.

Let R be a connected non empty poset. One can verify that FinPoset R is non empty.

Next we state the proposition

(40) Let R be a connected non empty poset and a, b be elements of FinPoset R. Then $\langle a, b\rangle \in$ the internal relation of FinPoset R if and only if there exist elements x, y of Fin (the carrier of R) such that $a = x$ but $b = y$ but $x = \emptyset$ or $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax $x \neq$ PosetMax y and $\langle \text{PosetMax}\,x, \text{PosetMax}\,y\rangle \in$ the internal relation of R or $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax $x =$ PosetMax y and $\langle x \setminus \{\text{PosetMax}\,x\}, y \setminus \{\text{PosetMax}\,y\}\rangle \in \text{FinOrd}\,R$.

Let R be a connected non empty poset. Note that FinPoset R is connected.

Let R be a connected non empty relational structure and let C be a non empty set. Let us assume that R is well founded and $C \subseteq$ the carrier of R. The functor MinElement(C,R) yielding an element of R is defined by:

(Def. 19) MinElement$(C,R) \in C$ and MinElement(C,R) is minimal w.r.t. C, the internal relation of R.

Let R be a non empty relational structure, let s be a sequence of R, and let j be a natural number. The functor SeqShift(s, j) yielding a sequence of R is defined by:

(Def. 20) For every natural number i holds $(\mathrm{SeqShift}(s, j))(i) = s(i + j)$.

We now state two propositions:

(41) Let R be a non empty relational structure, s be a sequence of R, and j be a natural number. If s is descending, then SeqShift(s, j) is descending.

(42) For every connected non empty poset R such that R is well founded holds FinPoset R is well founded.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[4] Grzegorz Bancerek. The well ordering relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/wellord1.html`.

[5] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[6] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_0.html`.

[7] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[8] Thomas Becker and Volker Weispfenning. *Gröbner Bases: A Computational Approach to Commutative Algebra*. Springer-Verlag, New York, Berlin, 1993.

[9] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[10] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[11] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[12] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[13] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseqop.html`.

[14] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_4.html`.

[15] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rvsum_1.html`.

[16] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[17] Adam Grabowski. Auxiliary and approximating relations. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_4.html`.

[18] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_1.html`.

[19] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[20] Andrzej Kondracki. The Chinese Remainder Theorem. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/wsierp_1.html`.

[21] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seqm_3.html`.

[22] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/rfinseq.html`.

[23] Gilbert Lee and Piotr Rudnicki. Dickson's lemma. *Journal of Formalized Mathematics*, 14, 2002. `http://mizar.org/JFM/Vol14/dickson.html`.

[24] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pralg_1.html`.

[25] Beata Madras. On the concept of the triangulation. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/triang_1.html`.

[26] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/pre_circ.html`.

[27] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/binarith.html`.

[28] Jan Popiołek. Real normed space. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/normsp_1.html`.

[29] Piotr Rudnicki and Andrzej Trybulec. On same equivalents of well-foundedness. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/wellfnd1.html`.

[30] Piotr Rudnicki and Andrzej Trybulec. Multivariate polynomials with arbitrary number of variables. *Journal of Formalized Mathematics*, 11, 1999. `http://mizar.org/JFM/Vol11/polynom1.html`.

[31] Christoph Schwarzweller and Andrzej Trybulec. The evaluation of multivariate polynomials. *Journal of Formalized Mathematics*, 12, 2000. `http://mizar.org/JFM/Vol12/polynom2.html`.

[32] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/domain_1.html`.

[33] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[34] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/mcart_1.html`.

[35] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pboole.html`.

[36] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[37] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finsub_1.html`.

[38] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/orders_1.html`.

[39] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[40] Wojciech A. Trybulec and Grzegorz Bancerek. Kuratowski - Zorn lemma. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/orders_2.html`.

[41] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[42] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[43] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

[44] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_2.html`.

Received March 12, 2002

Published January 2, 2004