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# Asymptotic Notation. Part II: Examples and Problems[1]

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**Summary.** The widely used textbook by Brassard and Bratley [4] includes a chapter devoted to asymptotic notation (Chapter 3, pp. 79–97). We have attempted to test how suitable the current version of Mizar is for recording this type of material in its entirety. This article is a follow-up to [13] in which we introduced the basic notions and general theory. This article presents a Mizar formalization of examples and solutions to problems from Chapter 3 of [4] (some of the examples and solved problems are also in [13]). Not all problems have been solved as some required solutions not amenable for formalization.

MML Identifier: `ASYMPT_1`.

WWW: `http://mizar.org/JFM/Vol11/asympt_1.html`

The articles [19], [25], [2], [21], [8], [5], [6], [20], [23], [1], [11], [9], [26], [14], [16], [17], [12], [15], [22], [10], [18], [3], [7], [24], and [13] provide the notation and terminology for this paper.

## 1. Examples from the Text

Let us note that every element of $\mathbb{N}$ is non negative.

We follow the rules: $c$, $e$ denote real numbers, $k$, $n$, $m$, $N$, $n_1$, $M$ denote natural numbers, and $x$ denotes a set.

The following propositions are true:

(1) Let $t$, $t_1$ be sequences of real numbers. Suppose that

(i) $t(0) = 0$,

(ii) for every $n$ such that $n > 0$ holds $t(n) = (12 \cdot n^3 \cdot \log_2 n - 5 \cdot n^{\mathbf{2}}) + (\log_2 n)^{\mathbf{2}} + 36$,

(iii) $t_1(0) = 0$, and

(iv) for every $n$ such that $n > 0$ holds $t_1(n) = n^3 \cdot \log_2 n$.

Then there exist eventually-positive sequences $s$, $s_1$ of real numbers such that $s = t$ and $s_1 = t_1$ and $s \in O(s_1)$.

(2) Let $a$, $b$ be logbase real numbers and $f$, $g$ be sequences of real numbers. Suppose $a > 1$ and $b > 1$ and $f(0) = 0$ and for every $n$ such that $n > 0$ holds $f(n) = \log_a n$ and $g(0) = 0$ and for every $n$ such that $n > 0$ holds $g(n) = \log_b n$. Then there exist eventually-positive sequences $s$, $s_1$ of real numbers such that $s = f$ and $s_1 = g$ and $O(s) = O(s_1)$.

Let $a$, $b$, $c$ be real numbers. The functor $\{a^{b \cdot n + c}\}_{n \in \mathbb{N}}$ yielding a sequence of real numbers is defined by:

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(Def. 1) $(\{a^{b \cdot n+c)}\}_{n \in \mathbb{N}})(n) = a^{b \cdot n+c}$.

Let $a$ be a positive real number and let $b$, $c$ be real numbers. One can verify that $\{a^{b \cdot n+c)}\}_{n \in \mathbb{N}}$ is eventually-positive.

One can prove the following proposition

(3) For all positive real numbers $a$, $b$ such that $a < b$ holds $\{b^{1 \cdot n+0)}\}_{n \in \mathbb{N}} \notin O(\{a^{1 \cdot n+0)}\}_{n \in \mathbb{N}})$.

The sequence $\{\log_2 n\}_{n \in \mathbb{N}}$ of real numbers is defined as follows:

(Def. 2) $\{\log_2 n\}_{n \in \mathbb{N}}(0) = 0$ and for every $n$ such that $n > 0$ holds $\{\log_2 n\}_{n \in \mathbb{N}}(n) = \log_2 n$.

Let $a$ be a real number. The functor $\{n^a\}_{n \in \mathbb{N}}$ yields a sequence of real numbers and is defined by:

(Def. 3) $\{n^a\}_{n \in \mathbb{N}}(0) = 0$ and for every $n$ such that $n > 0$ holds $\{n^a\}_{n \in \mathbb{N}}(n) = n^a$.

Let us note that $\{\log_2 n\}_{n \in \mathbb{N}}$ is eventually-positive.

Let $a$ be a real number. Observe that $\{n^a\}_{n \in \mathbb{N}}$ is eventually-positive.

We now state several propositions:

(4) Let $f$, $g$ be eventually-nonnegative sequences of real numbers. Then $O(f) \subseteq O(g)$ and $O(f) \neq O(g)$ if and only if $f \in O(g)$ and $f \notin \Omega(g)$.

(5) $O(\{\log_2 n\}_{n \in \mathbb{N}}) \subseteq O(\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}})$ and $O(\{\log_2 n\}_{n \in \mathbb{N}}) \neq O(\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}})$.

(6) $\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}} \in \Omega(\{\log_2 n\}_{n \in \mathbb{N}})$ and $\{\log_2 n\}_{n \in \mathbb{N}} \notin \Omega(\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}})$.

(7) For every sequence $f$ of real numbers and for every natural number $k$ such that for every $n$ holds $f(n) = \sum_{\kappa=0}^{n}(\{n^k\}_{n \in \mathbb{N}})(\kappa)$ holds $f \in \Theta(\{n^{(k+1)}\}_{n \in \mathbb{N}})$.

(8) Let $f$ be a sequence of real numbers. Suppose $f(0) = 0$ and for every $n$ such that $n > 0$ holds $f(n) = n^{\log_2 n}$. Then there exists an eventually-positive sequence $s$ of real numbers such that $s = f$ and $s$ is not smooth.

Let $b$ be a real number. The functor $\{b\}_{n \in \mathbb{N}}$ yielding a sequence of real numbers is defined as follows:

(Def. 4) $\{b\}_{n \in \mathbb{N}} = \mathbb{N} \longmapsto b$.

Let us note that $\{1\}_{n \in \mathbb{N}}$ is eventually-nonnegative.

We now state the proposition

(9) Let $f$ be an eventually-nonnegative sequence of real numbers. Then there exists a non empty set $F$ of functions from $\mathbb{N}$ to $\mathbb{R}$ such that $F = \{\{n^1\}_{n \in \mathbb{N}}\}$ and $f \in F^{O(\{1\}_{n \in \mathbb{N}})}$ iff there exist $N$, $c$, $k$ such that $c > 0$ and for every $n$ such that $n \geq N$ holds $1 \leq f(n)$ and $f(n) \leq c \cdot \{n^k\}_{n \in \mathbb{N}}(n)$.

## 2. Problem 3.1

We now state the proposition

(10) For every sequence $f$ of real numbers such that for every $n$ holds $f(n) = (3 \cdot 10^6 - 18 \cdot 10^3 \cdot n) + 27 \cdot n^{\mathbf{2}}$ holds $f \in O(\{n^2\}_{n \in \mathbb{N}})$.

## 3. PROBLEM 3.5

We now state three propositions:

(11) $\{n^2\}_{n\in\mathbb{N}} \in O(\{n^3\}_{n\in\mathbb{N}})$.

(12) $\{n^2\}_{n\in\mathbb{N}} \notin \Omega(\{n^3\}_{n\in\mathbb{N}})$.

(13) There exists an eventually-positive sequence $s$ of real numbers such that $s = \{2^{1\cdot n+1)}\}_{n\in\mathbb{N}}$ and $\{2^{1\cdot n+0)}\}_{n\in\mathbb{N}} \in \Theta(s)$.

Let $a$ be a natural number. The functor $\{(n+a)!\}_{n\in\mathbb{N}}$ yielding a sequence of real numbers is defined by:

(Def. 5) $\{(n+a)!\}_{n\in\mathbb{N}}(n) = (n+a)!$.

Let $a$ be a natural number. Note that $\{(n+a)!\}_{n\in\mathbb{N}}$ is eventually-positive.
Next we state the proposition

(14) $\{(n+0)!\}_{n\in\mathbb{N}} \notin \Theta(\{(n+1)!\}_{n\in\mathbb{N}})$.

## 4. PROBLEM 3.6

We now state the proposition

(15) For every sequence $f$ of real numbers such that $f \in O(\{n^1\}_{n\in\mathbb{N}})$ holds $f\,f \in O(\{n^2\}_{n\in\mathbb{N}})$.

## 5. PROBLEM 3.7

One can prove the following proposition

(16) There exists an eventually-positive sequence $s$ of real numbers such that $s = \{2^{1\cdot n+0)}\}_{n\in\mathbb{N}}$ and $2\,\{n^1\}_{n\in\mathbb{N}} \in O(\{n^1\}_{n\in\mathbb{N}})$ and $\{2^{2\cdot n+0)}\}_{n\in\mathbb{N}} \notin O(s)$.

## 6. PROBLEM 3.8

We now state the proposition

(17) If $\log_2 3 < \frac{159}{100}$, then $\{n^{(\log_2 3)}\}_{n\in\mathbb{N}} \in O(\{n^{(\frac{159}{100})}\}_{n\in\mathbb{N}})$ and $\{n^{(\log_2 3)}\}_{n\in\mathbb{N}} \notin \Omega(\{n^{(\frac{159}{100})}\}_{n\in\mathbb{N}})$ and $\{n^{(\log_2 3)}\}_{n\in\mathbb{N}} \notin \Theta(\{n^{(\frac{159}{100})}\}_{n\in\mathbb{N}})$.

## 7. PROBLEM 3.11

Next we state the proposition

(18) Let $f$, $g$ be sequences of real numbers. Suppose for every $n$ holds $f(n) = n \bmod 2$ and for every $n$ holds $g(n) = (n+1) \bmod 2$. Then there exist eventually-nonnegative sequences $s$, $s_1$ of real numbers such that $s = f$ and $s_1 = g$ and $s \notin O(s_1)$ and $s_1 \notin O(s)$.

## 8. PROBLEM 3.19

We now state two propositions:

(19) For all eventually-nonnegative sequences $f$, $g$ of real numbers holds $O(f) = O(g)$ iff $f \in \Theta(g)$.

(20) For all eventually-nonnegative sequences $f$, $g$ of real numbers holds $f \in \Theta(g)$ iff $\Theta(f) = \Theta(g)$.

## 9. Problem 3.21

One can prove the following propositions:

(21) Let $e$ be a real number and $f$ be a sequence of real numbers. Suppose $0 < e$ and $f(0) = 0$ and for every $n$ such that $n > 0$ holds $f(n) = n \cdot \log_2 n$. Then there exists an eventually-positive sequence $s$ of real numbers such that $s = f$ and $O(s) \subseteq O(\{n^{(1+e)}\}_{n \in \mathbb{N}})$ and $O(s) \neq O(\{n^{(1+e)}\}_{n \in \mathbb{N}})$.

(22) Let $e$ be a real number and $g$ be a sequence of real numbers. Suppose $0 < e$ and $e < 1$ and $g(0) = 0$ and $g(1) = 0$ and for every $n$ such that $n > 1$ holds $g(n) = \frac{n^2}{\log_2 n}$. Then there exists an eventually-positive sequence $s$ of real numbers such that $s = g$ and $O(\{n^{(1+e)}\}_{n \in \mathbb{N}}) \subseteq O(s)$ and $O(\{n^{(1+e)}\}_{n \in \mathbb{N}}) \neq O(s)$.

(23) Let $f$ be a sequence of real numbers. Suppose $f(0) = 0$ and $f(1) = 0$ and for every $n$ such that $n > 1$ holds $f(n) = \frac{n^2}{\log_2 n}$. Then there exists an eventually-positive sequence $s$ of real numbers such that $s = f$ and $O(s) \subseteq O(\{n^8\}_{n \in \mathbb{N}})$ and $O(s) \neq O(\{n^8\}_{n \in \mathbb{N}})$.

(24) Let $g$ be a sequence of real numbers. Suppose that for every $n$ holds $g(n) = ((n^{\mathbf{2}} - n) + 1)^4$. Then there exists an eventually-positive sequence $s$ of real numbers such that $s = g$ and $O(\{n^8\}_{n \in \mathbb{N}}) = O(s)$.

(25) Let $e$ be a real number. Suppose $0 < e$ and $e < 1$. Then there exists an eventually-positive sequence $s$ of real numbers such that $s = \{1 + e^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $O(\{n^8\}_{n \in \mathbb{N}}) \subseteq O(s)$ and $O(\{n^8\}_{n \in \mathbb{N}}) \neq O(s)$.

## 10. Problem 3.22

One can prove the following propositions:

(26) Let $f$, $g$ be sequences of real numbers. Suppose $f(0) = 0$ and for every $n$ such that $n > 0$ holds $f(n) = n^{\log_2 n}$ and $g(0) = 0$ and for every $n$ such that $n > 0$ holds $g(n) = n^{\sqrt{n}}$. Then there exist eventually-positive sequences $s$, $s_1$ of real numbers such that $s = f$ and $s_1 = g$ and $O(s) \subseteq O(s_1)$ and $O(s) \neq O(s_1)$.

(27) Let $f$ be a sequence of real numbers. Suppose $f(0) = 0$ and for every $n$ such that $n > 0$ holds $f(n) = n^{\sqrt{n}}$. Then there exist eventually-positive sequences $s$, $s_1$ of real numbers such that $s = f$ and $s_1 = \{2^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $O(s) \subseteq O(s_1)$ and $O(s) \neq O(s_1)$.

(28) There exist eventually-positive sequences $s$, $s_1$ of real numbers such that $s = \{2^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $s_1 = \{2^{1 \cdot n + 1)}\}_{n \in \mathbb{N}}$ and $O(s) = O(s_1)$.

(29) There exist eventually-positive sequences $s$, $s_1$ of real numbers such that $s = \{2^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $s_1 = \{2^{2 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $O(s) \subseteq O(s_1)$ and $O(s) \neq O(s_1)$.

(30) There exists an eventually-positive sequence $s$ of real numbers such that $s = \{2^{2 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $O(s) \subseteq O(\{(n+0)!\}_{n \in \mathbb{N}})$ and $O(s) \neq O(\{(n+0)!\}_{n \in \mathbb{N}})$.

(31) $O(\{(n+0)!\}_{n \in \mathbb{N}}) \subseteq O(\{(n+1)!\}_{n \in \mathbb{N}})$ and $O(\{(n+0)!\}_{n \in \mathbb{N}}) \neq O(\{(n+1)!\}_{n \in \mathbb{N}})$.

(32) Let $g$ be a sequence of real numbers. Suppose $g(0) = 0$ and for every $n$ such that $n > 0$ holds $g(n) = n^n$. Then there exists an eventually-positive sequence $s$ of real numbers such that $s = g$ and $O(\{(n+1)!\}_{n \in \mathbb{N}}) \subseteq O(s)$ and $O(\{(n+1)!\}_{n \in \mathbb{N}}) \neq O(s)$.

## 11. Problem 3.23

One can prove the following proposition

(33) Let given $n$. Suppose $n \geq 1$. Let $f$ be a sequence of real numbers and $k$ be a natural number. If for every $n$ holds $f(n) = \sum_{\kappa=0}^{n}(\{n^k\}_{n \in \mathbb{N}})(\kappa)$, then $f(n) \geq \frac{n^{k+1}}{k+1}$.

## 12. Problem 3.24

We now state the proposition

(34) Let $f$, $g$ be sequences of real numbers. Suppose $g(0) = 0$ and for every $n$ such that $n > 0$ holds $g(n) = n \cdot \log_2 n$ and for every $n$ holds $f(n) = \log_2(n!)$. Then there exists an eventually-nonnegative sequence $s$ of real numbers such that $s = g$ and $f \in \Theta(s)$.

## 13. Problem 3.26

One can prove the following proposition

(35) Let $f$ be an eventually-nondecreasing eventually-nonnegative sequence of real numbers and $t$ be a sequence of real numbers. Suppose that for every $n$ holds if $n \bmod 2 = 0$, then $t(n) = 1$ and if $n \bmod 2 = 1$, then $t(n) = n$. Then $t \notin \Theta(f)$.

## 14. Problem 3.28

Let $f$ be a function from $\mathbb{N}$ into $\mathbb{R}^*$ and let $n$ be a natural number. Then $f(n)$ is a finite sequence of elements of $\mathbb{R}$.

Let $n$ be a natural number and let $a$, $b$ be positive real numbers. The functor $\mathrm{Prob28}(n,a,b)$ yielding a real number is defined by:

(Def. 6)(i) $\mathrm{Prob28}(n,a,b) = 0$ if $n = 0$,

(ii) there exists a natural number $l$ and there exists a function $p_{28}$ from $\mathbb{N}$ into $\mathbb{R}^*$ such that $l+1 = n$ and $\mathrm{Prob28}(n,a,b) = p_{28}(l)_n$ and $p_{28}(0) = \langle a \rangle$ and for every natural number $n$ there exists a natural number $n_1$ such that $n_1 = \lceil \frac{n+1+1}{2} \rceil$ and $p_{28}(n+1) = p_{28}(n) \frown \langle 4 \cdot p_{28}(n)_{n_1} + b \cdot (n+1+1) \rangle$, otherwise.

Let $a$, $b$ be positive real numbers. The functor $\{\mathrm{Prob28}(n,a,b)\}_{n \in \mathbb{N}}$ yields a sequence of real numbers and is defined as follows:

(Def. 7) $(\{\mathrm{Prob28}(n,a,b)\}_{n \in \mathbb{N}})(n) = \mathrm{Prob28}(n,a,b)$.

We now state the proposition

(36) For all positive real numbers $a$, $b$ holds $\{\mathrm{Prob28}(n,a,b)\}_{n \in \mathbb{N}}$ is eventually-nondecreasing.

## 15. Problem 3.30

The non empty subset $\{2^n : n \in \mathbb{N}\}$ of $\mathbb{N}$ is defined by:

(Def. 8) $\{2^n : n \in \mathbb{N}\} = \{2^n : n$ ranges over natural numbers$\}$.

We now state three propositions:

(37) Let $f$ be a sequence of real numbers. Suppose that for every $n$ holds if $n \in \{2^n : n \in \mathbb{N}\}$, then $f(n) = n$ and if $n \notin \{2^n : n \in \mathbb{N}\}$, then $f(n) = 2^n$. Then $f \in \Theta(\{n^1\}_{n \in \mathbb{N}} | \{2^n : n \in \mathbb{N}\})$ and $f \notin \Theta(\{n^1\}_{n \in \mathbb{N}})$ and $\{n^1\}_{n \in \mathbb{N}}$ is smooth and $f$ is not eventually-nondecreasing.

(38) Let $f$, $g$ be sequences of real numbers. Suppose $f(0) = 0$ and for every $n$ such that $n > 0$ holds $f(n) = n^{2^{\lfloor \log_2 n \rfloor}}$ and $g(0) = 0$ and for every $n$ such that $n > 0$ holds $g(n) = n^n$. Then there exists an eventually-positive sequence $s$ of real numbers such that

(i) $s = g$,

(ii) $f \in \Theta(s | \{2^n : n \in \mathbb{N}\})$,

(iii) $f \notin \Theta(s)$,

(iv) $f$ is eventually-nondecreasing,

(v) $s$ is eventually-nondecreasing, and

(vi) $s$ is not smooth w.r.t. 2.

(39) Let $g$ be a sequence of real numbers. Suppose that for every $n$ holds if $n \in \{2^n : n \in \mathbb{N}\}$, then $g(n) = n$ and if $n \notin \{2^n : n \in \mathbb{N}\}$, then $g(n) = n^2$. Then there exists an eventually-positive sequence $s$ of real numbers such that $s = g$ and $\{n^1\}_{n \in \mathbb{N}} \in \Theta(s | \{2^n : n \in \mathbb{N}\})$ and $\{n^1\}_{n \in \mathbb{N}} \notin \Theta(s)$ and $s_2 \in O(s)$ and $\{n^1\}_{n \in \mathbb{N}}$ is eventually-nondecreasing and $s$ is not eventually-nondecreasing.

## 16. Problem 3.31

Let $x$ be a natural number. The functor $x_{¡}$ yielding a natural number is defined as follows:

(Def. 9)(i) There exists $n$ such that $n! \leq x$ and $x < (n+1)!$ and $x_{¡} = n!$ if $x \neq 0$,

(ii) $x_{¡} = 0$, otherwise.

The following proposition is true

(40) Let $f$ be a sequence of real numbers. Suppose that for every $n$ holds $f(n) = n_{¡}$. Then there exists an eventually-positive sequence $s$ of real numbers such that $s = f$ and $f$ is eventually-nondecreasing and for every $n$ holds $f(n) \leq \{n^1\}_{n \in \mathbb{N}}(n)$ and $s$ is not smooth.

## 17. Problem 3.34

Let us observe that $\{n^1\}_{n \in \mathbb{N}} - \{1\}_{n \in \mathbb{N}}$ is eventually-positive.

We now state the proposition

(41) $\Theta(\{n^1\}_{n \in \mathbb{N}} - \{1\}_{n \in \mathbb{N}}) + \Theta(\{n^1\}_{n \in \mathbb{N}}) = \Theta(\{n^1\}_{n \in \mathbb{N}})$.

## 18. Problem 3.35

The following proposition is true

(42) There exists a non empty set $F$ of functions from $\mathbb{N}$ to $\mathbb{R}$ such that $F = \{\{n^1\}_{n \in \mathbb{N}}\}$ and for every $n$ holds $\{n^{(-1)}\}_{n \in \mathbb{N}}(n) \leq \{n^1\}_{n \in \mathbb{N}}(n)$ and $\{n^{(-1)}\}_{n \in \mathbb{N}} \notin F^{O(\{1\}_{n \in \mathbb{N}})}$.

## 19. Addition

One can prove the following proposition

(43) Let $c$ be a non negative real number and $x$, $f$ be eventually-nonnegative sequences of real numbers. Given $e$, $N$ such that $e > 0$ and for every $n$ such that $n \geq N$ holds $f(n) \geq e$. If $x \in O(c + f)$, then $x \in O(f)$.

## 20. Potentatially Useful

We now state a number of propositions:

(44) $2^2 = 4$.

(45) $2^3 = 8$.

(46) $2^4 = 16$.

(47) $2^5 = 32$.

(48) $2^6 = 64$.

(49) $2^{12} = 4096$.

(50) For every $n$ such that $n \geq 3$ holds $n^{\mathbf{2}} > 2 \cdot n + 1$.

(51) For every $n$ such that $n \geq 10$ holds $2^{n-1} > (2 \cdot n)^{\mathbf{2}}$.

(52) For every $n$ such that $n \geq 9$ holds $(n+1)^6 < 2 \cdot n^6$.

(53) For every $n$ such that $n \geq 30$ holds $2^n > n^6$.

(54) For every real number $x$ such that $x > 9$ holds $2^x > (2 \cdot x)^{\mathbf{2}}$.

(55) There exists $N$ such that for every $n$ such that $n \geq N$ holds $\sqrt{n} - \log_2 n > 1$.

(56) For all real numbers $a$, $b$, $c$ such that $a > 0$ and $c > 0$ and $c \neq 1$ holds $a^b = c^{b \cdot \log_c a}$.

(57) $(4+1)! = 120$.

(58) $5^5 = 3125$.

(59) $4^4 = 256$.

(60) For every $n$ holds $(n^{\mathbf{2}} - n) + 1 > 0$.

(61) For every $n$ such that $n \geq 2$ holds $n! > 1$.

(62) For all $n_1$, $n$ such that $n \leq n_1$ holds $n! \leq n_1!$.

(63) For every $k$ such that $k \geq 1$ there exists $n$ such that $n! \leq k$ and $k < (n+1)!$ and for every $m$ such that $m! \leq k$ and $k < (m+1)!$ holds $m = n$.

(64) For every $n$ such that $n \geq 2$ holds $\lceil \frac{n}{2} \rceil < n$.

(65) For every $n$ such that $n \geq 3$ holds $n! > n$.

(67)[1] For every $n$ such that $n \geq 2$ holds $2^n > n+1$.

(68) Let $a$ be a logbase real number and $f$ be a sequence of real numbers. Suppose $a > 1$ and $f(0) = 0$ and for every $n$ such that $n > 0$ holds $f(n) = \log_a n$. Then $f$ is eventually-positive.

(69) For all eventually-nonnegative sequences $f$, $g$ of real numbers holds $f \in O(g)$ and $g \in O(f)$ iff $O(f) = O(g)$.

(70) For all real numbers $a$, $b$, $c$ such that $0 < a$ and $a \leq b$ and $c \geq 0$ holds $a^c \leq b^c$.

(71) For every $n$ such that $n \geq 4$ holds $2 \cdot n + 3 < 2^n$.

(72) For every $n$ such that $n \geq 6$ holds $(n+1)^{\mathbf{2}} < 2^n$.

(73) For every real number $c$ such that $c > 6$ holds $c^{\mathbf{2}} < 2^c$.

(74) Let $e$ be a positive real number and $f$ be a sequence of real numbers. Suppose $f(0) = 0$ and for every $n$ such that $n > 0$ holds $f(n) = \log_2(n^e)$. Then $f/\{n^e\}_{n \in \mathbb{N}}$ is convergent and $\lim(f/\{n^e\}_{n \in \mathbb{N}}) = 0$.

(75) For every real number $e$ such that $e > 0$ holds $\{\log_2 n\}_{n \in \mathbb{N}}/\{n^e\}_{n \in \mathbb{N}}$ is convergent and $\lim(\{\log_2 n\}_{n \in \mathbb{N}}/\{n^e\}_{n \in \mathbb{N}}) = 0$.

(76) For every sequence $f$ of real numbers and for every $N$ such that for every $n$ such that $n \leq N$ holds $f(n) \geq 0$ holds $\sum_{\kappa=0}^{N} f(\kappa) \geq 0$.

(77) For all sequences $f$, $g$ of real numbers and for every $N$ such that for every $n$ such that $n \leq N$ holds $f(n) \leq g(n)$ holds $\sum_{\kappa=0}^{N} f(\kappa) \leq \sum_{\kappa=0}^{N} g(\kappa)$.

(78) Let $f$ be a sequence of real numbers and $b$ be a real number. Suppose $f(0) = 0$ and for every $n$ such that $n > 0$ holds $f(n) = b$. Let $N$ be a natural number. Then $\sum_{\kappa=0}^{N} f(\kappa) = b \cdot N$.

(79) For all sequences $f$, $g$ of real numbers and for all natural numbers $N$, $M$ holds $\sum_{\kappa=N+1}^{M} f(\kappa) + f(N+1) = \sum_{\kappa=N+1+1}^{M} f(\kappa)$.

[1] The proposition (66) has been removed.

(80) Let $f$, $g$ be sequences of real numbers, $M$ be a natural number, and given $N$. Suppose $N \geq M+1$. If for every $n$ such that $M+1 \leq n$ and $n \leq N$ holds $f(n) \leq g(n)$, then $\sum_{\kappa=N+1}^{M} f(\kappa) \leq \sum_{\kappa=N+1}^{M} g(\kappa)$.

(81) For every $n$ holds $\lceil \frac{n}{2} \rceil \leq n$.

(82) Let $f$ be a sequence of real numbers, $b$ be a real number, and $N$ be a natural number. Suppose $f(0)=0$ and for every $n$ such that $n>0$ holds $f(n)=b$. Let $M$ be a natural number. Then $\sum_{\kappa=N+1}^{M} f(\kappa) = b \cdot (N-M)$.

(83) Let $f$, $g$ be sequences of real numbers, $N$ be a natural number, and $c$ be a real number. Suppose $f$ is convergent and $\lim f = c$ and for every $n$ such that $n \geq N$ holds $f(n)=g(n)$. Then $g$ is convergent and $\lim g = c$.

(84) For every $n$ such that $n \geq 1$ holds $(n^{\mathbf{2}} - n) + 1 \leq n^{\mathbf{2}}$.

(85) For every $n$ such that $n \geq 1$ holds $n^{\mathbf{2}} \leq 2 \cdot ((n^{\mathbf{2}} - n) + 1)$.

(86) For every real number $e$ such that $0 < e$ and $e < 1$ there exists $N$ such that for every $n$ such that $n \geq N$ holds $n \cdot \log_2(1+e) - 8 \cdot \log_2 n > 8 \cdot \log_2 n$.

(87) For every $n$ such that $n \geq 10$ holds $\frac{2^{2 \cdot n}}{n!} < \frac{1}{2^{n-9}}$.

(88) For every $n$ such that $n \geq 3$ holds $2 \cdot (n-2) \geq n-1$.

(89) For every real number $c$ such that $c \geq 0$ holds $c^{\frac{1}{2}} = \sqrt{c}$.

(90) There exists $N$ such that for every $n$ such that $n \geq N$ holds $n - \sqrt{n} \cdot \log_2 n > \frac{n}{2}$.

(91) For every sequence $s$ of real numbers such that for every $n$ holds $s(n) = (1+\frac{1}{n+1})^{n+1}$ holds $s$ is non-decreasing.

(92) For every $n$ such that $n \geq 1$ holds $(\frac{n+1}{n})^n \leq (\frac{n+2}{n+1})^{n+1}$.

(93) For all $k$, $n$ such that $k \leq n$ holds $\binom{n}{k} \geq \frac{\binom{n+1}{k}}{n+1}$.

(94) For every sequence $f$ of real numbers such that for every $n$ holds $f(n) = \log_2(n!)$ and for every $n$ holds $f(n) = \sum_{\kappa=0}^{n}(\{\log_2 n\}_{n \in \mathbb{N}})(\kappa)$.

(95) For every $n$ such that $n \geq 4$ holds $n \cdot \log_2 n \geq 2 \cdot n$.

(96) Let $a$, $b$ be positive real numbers. Then $\mathrm{Prob28}(0,a,b) = 0$ and $\mathrm{Prob28}(1,a,b) = a$ and for every $n$ such that $n \geq 2$ there exists $n_1$ such that $n_1 = \lceil \frac{n}{2} \rceil$ and $\mathrm{Prob28}(n,a,b) = 4 \cdot \mathrm{Prob28}(n_1,a,b) + b \cdot n$.

(97) For every $n$ such that $n \geq 2$ holds $n^{\mathbf{2}} > n+1$.

(98) For every $n$ such that $n \geq 1$ holds $2^{n+1} - 2^n > 1$.

(99) For every $n$ such that $n \geq 2$ holds $2^n - 1 \notin \{2^n : n \in \mathbb{N}\}$.

(100) For all $n$, $k$ such that $k \geq 1$ and $n! \leq k$ and $k < (n+1)!$ holds $k¡ = n!$.

(101) For all real numbers $a$, $b$, $c$ such that $a > 1$ and $b \geq a$ and $c \geq 1$ holds $\log_a c \geq \log_b c$.

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