JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

A Construction of Analytical Projective Space

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Summary. The collinearity structure denoted by ProjectiveSpace(V) is correlated with a given vector space V (over the field of Reals). It is a formalization of the standard construction of a projective space, where points are interpreted as equivalence classes of the relation of proportionality considered in the set of all non-zero vectors. Then the relation of collinearity corresponds to the relation of linear dependence of vectors. Several facts concerning vectors are proved, which correspond in this language to some classical axioms of projective geometry.

MML Identifier: ANPROJ_1.

WWW: http://mizar.org/JFM/Vol2/anproj_1.html

The articles [7], [2], [10], [3], [5], [4], [1], [6], [9], and [8] provide the notation and terminology for this paper.

We adopt the following convention: V denotes a real linear space, p, q, r, u, v, w, y, u_1, v_1, w_1 denote elements of V, and a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 denote real numbers.

Let V be a real linear space and let p be an element of V. We introduce p is a proper vector as a synonym of p is non-zero.

Let us consider V, p, q. We say that p and q are proportional if and only if:

(Def. 2)[1] There exist a, b such that $a \cdot p = b \cdot q$ and $a \neq 0$ and $b \neq 0$.

Let us notice that the predicate p and q are proportional is reflexive and symmetric.

We now state three propositions:

(5)[2] p and q are proportional iff there exists a such that $a \neq 0$ and $p = a \cdot q$.

(6) If p and u are proportional and u and q are proportional, then p and q are proportional.

(7) p and 0_V are proportional iff $p = 0_V$.

Let us consider V, u, v, w. We say that u, v and w are lineary dependent if and only if:

(Def. 3) There exist a, b, c such that $a \cdot u + b \cdot v + c \cdot w = 0_V$ but $a \neq 0$ or $b \neq 0$ or $c \neq 0$.

One can prove the following propositions:

(9)[3] Suppose that

(i) u and u_1 are proportional,

[1] The definition (Def. 1) has been removed.
[2] The propositions (1)–(4) have been removed.
[3] The proposition (8) has been removed.

(ii) v and v_1 are proportional,

(iii) w and w_1 are proportional, and

(iv) u, v and w are lineary dependent.

Then u_1, v_1 and w_1 are lineary dependent.

(10) Suppose u, v and w are lineary dependent. Then

(i) u, w and v are lineary dependent,

(ii) v, u and w are lineary dependent,

(iii) w, v and u are lineary dependent,

(iv) w, u and v are lineary dependent, and

(v) v, w and u are lineary dependent.

(11) Suppose v is a proper vector and w is a proper vector and v and w are not proportional. Then v, w and u are lineary dependent if and only if there exist a, b such that $u = a \cdot v + b \cdot w$.

(12) Suppose p and q are not proportional and $a_1 \cdot p + b_1 \cdot q = a_2 \cdot p + b_2 \cdot q$ and p is a proper vector and q is a proper vector. Then $a_1 = a_2$ and $b_1 = b_2$.

(13) If u, v and w are not lineary dependent and $a_1 \cdot u + b_1 \cdot v + c_1 \cdot w = a_2 \cdot u + b_2 \cdot v + c_2 \cdot w$, then $a_1 = a_2$ and $b_1 = b_2$ and $c_1 = c_2$.

(14) Suppose that

(i) p and q are not proportional,

(ii) $u = a_1 \cdot p + b_1 \cdot q$,

(iii) $v = a_2 \cdot p + b_2 \cdot q$,

(iv) $a_1 \cdot b_2 - a_2 \cdot b_1 = 0$,

(v) p is a proper vector, and

(vi) q is a proper vector.

Then u and v are proportional or $u = 0_V$ or $v = 0_V$.

(15) If $u = 0_V$ or $v = 0_V$ or $w = 0_V$, then u, v and w are lineary dependent.

(16) Suppose u and v are proportional or w and u are proportional or v and w are proportional. Then w, u and v are lineary dependent.

(17) Suppose u, v and w are not lineary dependent. Then

(i) u is a proper vector,

(ii) v is a proper vector,

(iii) w is a proper vector,

(iv) u and v are not proportional,

(v) v and w are not proportional, and

(vi) w and u are not proportional.

(18) If $p + q = 0_V$, then p and q are proportional.

(19) Suppose that

(i) p and q are not proportional,

(ii) p, q and u are lineary dependent,

(iii) p, q and v are lineary dependent,

(iv) p, q and w are lineary dependent,

(v) p is a proper vector, and

(vi) q is a proper vector.

Then u, v and w are lineary dependent.

(20) Suppose that

(i) u, v and w are not lineary dependent,

(ii) u, v and p are lineary dependent, and

(iii) v, w and q are lineary dependent.

Then there exists y such that u, w and y are lineary dependent and p, q and y are lineary dependent and y is a proper vector.

(21) Suppose p and q are not proportional and p is a proper vector and q is a proper vector. Let given u, v. Then there exists y such that

(i) y is a proper vector,

(ii) u, v and y are lineary dependent,

(iii) u and y are not proportional, and

(iv) v and y are not proportional.

(22) Suppose p, q and r are not lineary dependent. Let given u, v. Suppose u is a proper vector and v is a proper vector and u and v are not proportional. Then there exists y such that y is a proper vector and u, v and y are not lineary dependent.

(23) Suppose that u, v and q are lineary dependent and w, y and q are lineary dependent and u, w and p are lineary dependent and v, y and p are lineary dependent and u, y and r are lineary dependent and v, w and r are lineary dependent and p, q and r are lineary dependent and p is a proper vector and q is a proper vector and r is a proper vector. Then

(i) u, v and y are lineary dependent, or

(ii) u, v and w are lineary dependent, or

(iii) u, w and y are lineary dependent, or

(iv) v, w and y are lineary dependent.

In the sequel x, y, z are sets.

Let us consider V. The proper vectors of V is defined by:

(Def. 4) For every set u holds $u \in$ the proper vectors of V iff $u \neq 0_V$ and u is an element of V.

One can prove the following proposition

(26)[4] For every u holds $u \in$ the proper vectors of V iff u is a proper vector.

Let us consider V. The proportionality in V yielding an equivalence relation of the proper vectors of V is defined by the condition (Def. 5).

(Def. 5) Let given x, y. Then $\langle x, y \rangle \in$ the proportionality in V if and only if the following conditions are satisfied:

(i) $x \in$ the proper vectors of V,

(ii) $y \in$ the proper vectors of V, and

(iii) there exist elements u, v of V such that $x = u$ and $y = v$ and u and v are proportional.

The following propositions are true:

(28)[5] If $\langle x, y \rangle \in$ the proportionality in V, then x is an element of V and y is an element of V.

(29) $\langle u, v \rangle \in$ the proportionality in V if and only if u is a proper vector and v is a proper vector and u and v are proportional.

[4] The propositions (24) and (25) have been removed.

[5] The proposition (27) has been removed.

Let us consider V and let us consider v. The direction of v yields a subset of the proper vectors of V and is defined by:

(Def. 6) The direction of $v = [v]_{\text{the proportionality in } V}$.

Let us consider V. The projective points over V is defined by the condition (Def. 7).

(Def. 7) There exists a family Y of subsets of the proper vectors of V such that $Y =$ Classes(the proportionality in V) and the projective points over $V = Y$.

Let V be a non empty zero structure. Let us observe that V is trivial if and only if:

(Def. 8) For every element u of V holds $u = 0_V$.

Let us mention that there exists a real linear space which is strict and non trivial.

The following proposition is true

(33)[6] Let V be a real linear space. Then V is a non trivial real linear space if and only if there exists an element u of V such that $u \in$ the proper vectors of V.

We follow the rules: V is a non trivial real linear space and p, q, r, u, v, w are elements of V.

Let us consider V. Observe that the proper vectors of V is non empty and the projective points over V is non empty.

The following two propositions are true:

(34) If p is a proper vector, then the direction of p is an element of the projective points over V.

(35) Suppose p is a proper vector and q is a proper vector. Then the direction of $p =$ the direction of q if and only if p and q are proportional.

Let us consider V. The projective collinearity over V yielding a 3-ary relation of the projective points over V is defined by the condition (Def. 9).

(Def. 9) Let x, y, z be sets. Then $\langle x, y, z\rangle \in$ the projective collinearity over V if and only if there exist p, q, r such that $x =$ the direction of p and $y =$ the direction of q and $z =$ the direction of r and p is a proper vector and q is a proper vector and r is a proper vector and p, q and r are lineary dependent.

Let us consider V. The projective space over V yielding a strict collinearity structure is defined as follows:

(Def. 10) The projective space over $V = \langle$the projective points over V, the projective collinearity over $V\rangle$.

Let us consider V. Observe that the projective space over V is non empty.

Next we state four propositions:

(39)[7] Let given V. Then

(i) the carrier of the projective space over $V =$ the projective points over V, and

(ii) the collinearity relation of the projective space over $V =$ the projective collinearity over V.

(40) Suppose $\langle x, y, z\rangle \in$ the collinearity relation of the projective space over V. Then there exist p, q, r such that

$x =$ the direction of p and $y =$ the direction of q and $z =$ the direction of r and p is a proper vector and q is a proper vector and r is a proper vector and p, q and r are lineary dependent.

(41) Suppose u is a proper vector and v is a proper vector and w is a proper vector. Then $\langle$the direction of u, the direction of v, the direction of $w\rangle \in$ the collinearity relation of the projective space over V if and only if u, v and w are lineary dependent.

(42) x is an element of the projective space over V if and only if there exists u such that u is a proper vector and $x =$ the direction of u.

[6] The propositions (30)–(32) have been removed.

[7] The propositions (36)–(38) have been removed.

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Received June 15, 1990

Published January 2, 2004