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On a Mathematical Model of Programs

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Summary. We continue the work on mathematical modeling of hardware and software started in [11]. The main objective of this paper is the definition of a program. We start with the concept of partial product, i.e. the set of all partial functions f from I to $\bigcup_{i \in I} A_i$, fulfilling the condition $f.i \in A_i$ for $i \in dom f$. The computation and the result of a computation are defined in usual way. A finite partial state is called autonomic if the result of a computation starting with it does not depend on the remaining memory and an AMI is called programmable if it has a non empty autonomic partial finite state. We prove the consistency of the following set of properties of an AMI: data-oriented, halting, steady-programmed, realistic and programmable. For this purpose we define a trivial AMI. It has only the instruction counter and one instruction location. The only instruction of it is the halt instruction. A pre-program is a finite partial state that halts. We conclude with the definition of a program of a partial function F mapping the set of the finite partial states into itself. It is a finite partial state s such that for every finite partial state $s' \in dom F$ the result of any computation starting with $s+s'$ includes $F.s'$.

MML Identifier: `AMI_2`.

WWW: `http://mizar.org/JFM/Vol4/ami_2.html`

The articles [15], [14], [7], [20], [2], [17], [3], [21], [5], [6], [12], [13], [18], [1], [8], [16], [9], [10], [4], and [19] provide the notation and terminology for this paper.

In this paper x denotes a set and i, k denote natural numbers.

The element $\mathrm{Halt}_{\mathrm{SCM}}$ of $\mathbb{Z}_9$ is defined as follows:

(Def. 1) $\mathrm{Halt}_{\mathrm{SCM}} = 0$.

The subset $\text{Data-Loc}_{\mathrm{SCM}}$ of $\mathbb{N}$ is defined by:

(Def. 2) $\text{Data-Loc}_{\mathrm{SCM}} = \{2 \cdot k + 1\}$.

The subset $\text{Instr-Loc}_{\mathrm{SCM}}$ of $\mathbb{N}$ is defined by:

(Def. 3) $\text{Instr-Loc}_{\mathrm{SCM}} = \{2 \cdot k : k > 0\}$.

Let us observe that $\text{Data-Loc}_{\mathrm{SCM}}$ is non empty and $\text{Instr-Loc}_{\mathrm{SCM}}$ is non empty.

We use the following convention: I, J, K denote elements of $\mathbb{Z}_9$, a, a_1, a_2 denote elements of $\text{Instr-Loc}_{\mathrm{SCM}}$, and b, b_1, b_2, c, c_1 denote elements of $\text{Data-Loc}_{\mathrm{SCM}}$.

The subset $\mathrm{Instr}_{\mathrm{SCM}}$ of $[: \mathbb{Z}_9, (\bigcup\{\mathbb{Z}\} \cup \mathbb{N})^* :]$ is defined by:

(Def. 4) $\mathrm{Instr}_{\mathrm{SCM}} = \{\langle \mathrm{Halt}_{\mathrm{SCM}}, \emptyset \rangle\} \cup \{\langle J, \langle a \rangle \rangle : J = 6\} \cup \{\langle K, \langle a_1, b_1 \rangle \rangle : K \in \{7,8\}\} \cup \{\langle I, \langle b, c \rangle \rangle : I \in \{1,2,3,4,5\}\}$.

The following proposition is true

(2)[1] $\langle 0, \emptyset \rangle \in \mathrm{Instr}_{\mathrm{SCM}}$.

One can check that $\mathrm{Instr}_{\mathrm{SCM}}$ is non empty.
One can prove the following three propositions:

(3) $\langle 6, \langle a_2 \rangle \rangle \in \mathrm{Instr}_{\mathrm{SCM}}$.

(4) If $x \in \{7, 8\}$, then $\langle x, \langle a_2, b_2 \rangle \rangle \in \mathrm{Instr}_{\mathrm{SCM}}$.

(5) If $x \in \{1, 2, 3, 4, 5\}$, then $\langle x, \langle b_1, c_1 \rangle \rangle \in \mathrm{Instr}_{\mathrm{SCM}}$.

The function $\mathrm{OK}_{\mathrm{SCM}}$ from $\mathbb{N}$ into $\{\mathbb{Z}\} \cup \{\mathrm{Instr}_{\mathrm{SCM}}, \mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}\}$ is defined by:

(Def. 5) $\mathrm{OK}_{\mathrm{SCM}}(0) = \mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$ and for every natural number k holds $\mathrm{OK}_{\mathrm{SCM}}(2 \cdot k + 1) = \mathbb{Z}$ and $\mathrm{OK}_{\mathrm{SCM}}(2 \cdot k + 2) = \mathrm{Instr}_{\mathrm{SCM}}$.

One can prove the following propositions:

(6) $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}} \neq \mathbb{Z}$ and $\mathrm{Instr}_{\mathrm{SCM}} \neq \mathbb{Z}$ and $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}} \neq \mathrm{Instr}_{\mathrm{SCM}}$.

(7) $\mathrm{OK}_{\mathrm{SCM}}(i) = \mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$ iff $i = 0$.

(8) $\mathrm{OK}_{\mathrm{SCM}}(i) = \mathbb{Z}$ iff there exists k such that $i = 2 \cdot k + 1$.

(9) $\mathrm{OK}_{\mathrm{SCM}}(i) = \mathrm{Instr}_{\mathrm{SCM}}$ iff there exists k such that $i = 2 \cdot k + 2$.

A **SCM**-state is an element of $\prod(\mathrm{OK}_{\mathrm{SCM}})$.
Next we state several propositions:

(10) For every element a of $\mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$ holds $\mathrm{OK}_{\mathrm{SCM}}(a) = \mathbb{Z}$.

(11) For every element a of $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$ holds $\mathrm{OK}_{\mathrm{SCM}}(a) = \mathrm{Instr}_{\mathrm{SCM}}$.

(12) For every element a of $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$ and for every element t of $\mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$ holds $a \neq t$.

(13) $\pi_0 \prod(\mathrm{OK}_{\mathrm{SCM}}) = \mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$.

(14) For every element a of $\mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$ holds $\pi_a \prod(\mathrm{OK}_{\mathrm{SCM}}) = \mathbb{Z}$.

(15) For every element a of $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$ holds $\pi_a \prod(\mathrm{OK}_{\mathrm{SCM}}) = \mathrm{Instr}_{\mathrm{SCM}}$.

Let s be a **SCM**-state. The functor $\mathbf{IC}_s$ yielding an element of $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$ is defined by:

(Def. 6) $\mathbf{IC}_s = s(0)$.

Let s be a **SCM**-state and let u be an element of $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$. The functor $\mathrm{Chg}_{\mathrm{SCM}}(s, u)$ yields a **SCM**-state and is defined as follows:

(Def. 7) $\mathrm{Chg}_{\mathrm{SCM}}(s, u) = s +\cdot (0 \dot{\longmapsto} u)$.

We now state three propositions:

(16) For every **SCM**-state s and for every element u of $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$ holds $(\mathrm{Chg}_{\mathrm{SCM}}(s, u))(0) = u$.

(17) For every **SCM**-state s and for every element u of $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$ and for every element m_1 of $\mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$ holds $(\mathrm{Chg}_{\mathrm{SCM}}(s, u))(m_1) = s(m_1)$.

(18) For every **SCM**-state s and for all elements u, v of $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$ holds $(\mathrm{Chg}_{\mathrm{SCM}}(s, u))(v) = s(v)$.

Let s be a **SCM**-state, let t be an element of $\mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$, and let u be an integer. The functor $\mathrm{Chg}_{\mathrm{SCM}}(s, t, u)$ yields a **SCM**-state and is defined by:

[1] The proposition (1) has been removed.

(Def. 8) $\mathrm{Chg}_{\mathrm{SCM}}(s,t,u) = s+\cdot(t\dot{\longmapsto}u)$.

Next we state four propositions:

(19) For every **SCM**-state s and for every element t of Data-Loc$_{\mathrm{SCM}}$ and for every integer u holds $(\mathrm{Chg}_{\mathrm{SCM}}(s,t,u))(0) = s(0)$.

(20) For every **SCM**-state s and for every element t of Data-Loc$_{\mathrm{SCM}}$ and for every integer u holds $(\mathrm{Chg}_{\mathrm{SCM}}(s,t,u))(t) = u$.

(21) Let s be a **SCM**-state, t be an element of Data-Loc$_{\mathrm{SCM}}$, u be an integer, and m_1 be an element of Data-Loc$_{\mathrm{SCM}}$. If $m_1 \neq t$, then $(\mathrm{Chg}_{\mathrm{SCM}}(s,t,u))(m_1) = s(m_1)$.

(22) Let s be a **SCM**-state, t be an element of Data-Loc$_{\mathrm{SCM}}$, u be an integer, and v be an element of Instr-Loc$_{\mathrm{SCM}}$. Then $(\mathrm{Chg}_{\mathrm{SCM}}(s,t,u))(v) = s(v)$.

Let x be an element of Instr$_{\mathrm{SCM}}$. Let us assume that there exist elements m_1, m_2 of Data-Loc$_{\mathrm{SCM}}$ and I such that $x = \langle I, \langle m_1, m_2\rangle\rangle$. The functor $x\,\mathrm{address}_1$ yielding an element of Data-Loc$_{\mathrm{SCM}}$ is defined as follows:

(Def. 9) There exists a finite sequence f of elements of Data-Loc$_{\mathrm{SCM}}$ such that $f = x_{\mathbf{2}}$ and $x\,\mathrm{address}_1 = f_1$.

The functor $x\,\mathrm{address}_2$ yields an element of Data-Loc$_{\mathrm{SCM}}$ and is defined by:

(Def. 10) There exists a finite sequence f of elements of Data-Loc$_{\mathrm{SCM}}$ such that $f = x_{\mathbf{2}}$ and $x\,\mathrm{address}_2 = f_2$.

One can prove the following proposition

(23) For every element x of Instr$_{\mathrm{SCM}}$ and for all elements m_1, m_2 of Data-Loc$_{\mathrm{SCM}}$ and for every I such that $x = \langle I, \langle m_1, m_2\rangle\rangle$ holds $x\,\mathrm{address}_1 = m_1$ and $x\,\mathrm{address}_2 = m_2$.

Let x be an element of Instr$_{\mathrm{SCM}}$. Let us assume that there exist an element m_1 of Instr-Loc$_{\mathrm{SCM}}$ and I such that $x = \langle I, \langle m_1\rangle\rangle$. The functor $x\,\mathrm{address}_{\mathrm{j}}$ yields an element of Instr-Loc$_{\mathrm{SCM}}$ and is defined as follows:

(Def. 11) There exists a finite sequence f of elements of Instr-Loc$_{\mathrm{SCM}}$ such that $f = x_{\mathbf{2}}$ and $x\,\mathrm{address}_{\mathrm{j}} = f_1$.

One can prove the following proposition

(24) For every element x of Instr$_{\mathrm{SCM}}$ and for every element m_1 of Instr-Loc$_{\mathrm{SCM}}$ and for every I such that $x = \langle I, \langle m_1\rangle\rangle$ holds $x\,\mathrm{address}_{\mathrm{j}} = m_1$.

Let x be an element of Instr$_{\mathrm{SCM}}$. Let us assume that there exist an element m_1 of Instr-Loc$_{\mathrm{SCM}}$, an element m_2 of Data-Loc$_{\mathrm{SCM}}$, and I such that $x = \langle I, \langle m_1, m_2\rangle\rangle$. The functor $x\,\mathrm{address}_{\mathrm{j}}$ yields an element of Instr-Loc$_{\mathrm{SCM}}$ and is defined by:

(Def. 12) There exists an element m_1 of Instr-Loc$_{\mathrm{SCM}}$ and there exists an element m_2 of Data-Loc$_{\mathrm{SCM}}$ such that $\langle m_1, m_2\rangle = x_{\mathbf{2}}$ and $x\,\mathrm{address}_{\mathrm{j}} = \langle m_1, m_2\rangle_1$.

The functor $x\,\mathrm{address}_{\mathrm{c}}$ yielding an element of Data-Loc$_{\mathrm{SCM}}$ is defined by:

(Def. 13) There exists an element m_1 of Instr-Loc$_{\mathrm{SCM}}$ and there exists an element m_2 of Data-Loc$_{\mathrm{SCM}}$ such that $\langle m_1, m_2\rangle = x_{\mathbf{2}}$ and $x\,\mathrm{address}_{\mathrm{c}} = \langle m_1, m_2\rangle_2$.

Next we state the proposition

(25) Let x be an element of Instr$_{\mathrm{SCM}}$, m_1 be an element of Instr-Loc$_{\mathrm{SCM}}$, m_2 be an element of Data-Loc$_{\mathrm{SCM}}$, and given I. If $x = \langle I, \langle m_1, m_2\rangle\rangle$, then $x\,\mathrm{address}_{\mathrm{j}} = m_1$ and $x\,\mathrm{address}_{\mathrm{c}} = m_2$.

Let s be a **SCM**-state and let a be an element of $\text{Data-Loc}_{\text{SCM}}$. Note that $s(a)$ is integer.

Let D be a non empty set, let x, y be real numbers, and let a, b be elements of D. The functor $(x > y \to a, b)$ yields an element of D and is defined as follows:

(Def. 14) $$(x > y \to a, b) = \begin{cases} a, & \text{if } x > y, \\ b, & \text{otherwise.} \end{cases}$$

Let d be an element of $\text{Instr-Loc}_{\text{SCM}}$. The functor $\text{Next}(d)$ yields an element of $\text{Instr-Loc}_{\text{SCM}}$ and is defined by:

(Def. 15) $\text{Next}(d) = d + 2.$

Let x be an element of $\text{Instr}_{\text{SCM}}$ and let s be a **SCM**-state. The functor $\text{Exec-Res}_{\text{SCM}}(x, s)$ yields a **SCM**-state and is defined as follows:

(Def. 16) $$\text{Exec-Res}_{\text{SCM}}(x,s) = \begin{cases} \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\,\text{address}_1, s(x\,\text{address}_2)), \text{Next}(\mathbf{IC}_s)), & \text{if thereexistelements } m_1,\ m_2 \text{ of Data-Loc}_{\text{SCM}} \text{ such that } x = \langle\!\langle 1, \langle m_1, m_2 \rangle \rangle\!\rangle, \\ \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\,\text{address}_1, s(x\,\text{address}_1) + s(x\,\text{address}_2)), \text{Next}(\mathbf{IC}_s)), & \text{if thereexistelements } m_1,\ m_2 \text{ of Data-Loc}_{\text{SCM}} \text{ such that } x = \langle\!\langle 2, \langle m_1, m_2 \rangle \rangle\!\rangle, \\ \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\,\text{address}_1, s(x\,\text{address}_1) - s(x\,\text{address}_2)), \text{Next}(\mathbf{IC}_s)), & \text{if thereexistelements } m_1,\ m_2 \text{ of Data-Loc}_{\text{SCM}} \text{ such that } x = \langle\!\langle 3, \langle m_1, m_2 \rangle \rangle\!\rangle, \\ \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\,\text{address}_1, s(x\,\text{address}_1) \cdot s(x\,\text{address}_2)), \text{Next}(\mathbf{IC}_s)), & \text{if thereexistelements } m_1,\ m_2 \text{ of Data-Loc}_{\text{SCM}} \text{ such that } x = \langle\!\langle 4, \langle m_1, m_2 \rangle \rangle\!\rangle, \\ \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\,\text{address}_1, s(x\,\text{address}_1) \div s(x\,\text{address}_2)), x\,\text{address}_2, s(x\,\text{address}_1) \bmod s(x\,\text{address}_2)), \text{Next}(\mathbf{IC}_s)), & \text{if thereexistelements } m_1,\ m_2 \text{ of Data-Loc}_{\text{SCM}} \text{ such that } x = \langle\!\langle 5, \langle m_1, m_2 \rangle \rangle\!\rangle, \\ \text{Chg}_{\text{SCM}}(s, x\,\text{address}_{\text{j}}), & \text{if thereexists an element } m_1 \text{ of Instr-Loc}_{\text{SCM}} \text{ such that } x = \langle\!\langle 6, \langle m_1 \rangle \rangle\!\rangle, \\ \text{Chg}_{\text{SCM}}(s, (s(x\,\text{address}_{\text{c}}) = 0 \to x\,\text{address}_{\text{j}}, \text{Next}(\mathbf{IC}_s))), & \text{if thereexists an element } m_1 \text{ of Instr-Loc}_{\text{SCM}} \text{ and thereexists an element } m_2 \text{ of Data-Loc}_{\text{SCM}} \text{ such that } x = \langle\!\langle 7, \langle m_1, m_2 \rangle \rangle\!\rangle, \\ \text{Chg}_{\text{SCM}}(s, (s(x\,\text{address}_{\text{c}}) > 0 \to x\,\text{address}_{\text{j}}, \text{Next}(\mathbf{IC}_s))), & \text{if thereexists an element } m_1 \text{ of Instr-Loc}_{\text{SCM}} \text{ and thereexists an element } m_2 \text{ of Data-Loc}_{\text{SCM}} \text{ such that } x = \langle\!\langle 8, \langle m_1, m_2 \rangle \rangle\!\rangle, \\ s, & \text{otherwise.} \end{cases}$$

The function Exec_{SCM} from $\text{Instr}_{\text{SCM}}$ into $(\prod(\text{OK}_{\text{SCM}}))^{\prod(\text{OK}_{\text{SCM}})}$ is defined by:

(Def. 17) For every element x of $\text{Instr}_{\text{SCM}}$ and for every **SCM**-state y holds $\text{Exec}_{\text{SCM}}(x)(y) = \text{Exec-Res}_{\text{SCM}}(x, y)$.

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