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Examples of Category Structures

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Summary. We continue the formalization of the category theory.

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The articles [14], [6], [19], [20], [15], [3], [4], [2], [13], [1], [8], [7], [9], [5], [16], [12], [18], [17], [10], and [11] provide the notation and terminology for this paper.

1. Preliminaries

The following proposition is true

(1) For all sets X_1, X_2 and for all sets a_1, a_2 holds $[:X_1 \longmapsto a_1, X_2 \longmapsto a_2:] = [:X_1, X_2:] \longmapsto \langle a_1, a_2 \rangle$.

Let I be a set. One can check that $\mathbf{0}_I$ is function yielding.

Next we state two propositions:

(2) For all functions f, g holds $\curvearrowleft(g \cdot f) = g \cdot \curvearrowleft f$.

(3) For all functions f, g, h holds $\curvearrowleft(f \cdot [:g, h:]) = \curvearrowleft f \cdot [:h, g:]$.

Let f be a function yielding function. Observe that $\curvearrowleft f$ is function yielding.

Next we state the proposition

(4) Let I be a set and A, B, C be many sorted sets indexed by I. Suppose A is transformable to B. Let F be a many sorted function from A into B and G be a many sorted function from B into C. Then $G \circ F$ is a many sorted function from A into C.

Let I be a set and let A be a many sorted set indexed by $[:I, I:]$. Then $\curvearrowleft A$ is a many sorted set indexed by $[:I, I:]$.

Next we state the proposition

(5) Let I_1 be a set, I_2 be a non empty set, f be a function from I_1 into I_2, B, C be many sorted sets indexed by I_2, and G be a many sorted function from B into C. Then $G \cdot f$ is a many sorted function from $B \cdot f$ into $C \cdot f$.

Let I be a set, let A, B be many sorted sets indexed by $[:I, I:]$, and let F be a many sorted function from A into B. Then $\curvearrowleft F$ is a many sorted function from $\curvearrowleft A$ into $\curvearrowleft B$.

Next we state the proposition

(6) Let I_1, I_2 be non empty sets, M be a many sorted set indexed by $[: I_1, I_2 :]$, o_1 be an element of I_1, and o_2 be an element of I_2. Then $(\curvearrowleft M)(o_2, o_1) = M(o_1, o_2)$.

Let I_1 be a set and let f, g be many sorted functions indexed by I_1. Then $g \circ f$ is a many sorted function indexed by I_1.

2. An auxiliary notion

Let f, g be functions. The predicate $f \dot{\subseteq} g$ is defined as follows:

(Def. 1) $\operatorname{dom} f \subseteq \operatorname{dom} g$ and for every set i such that $i \in \operatorname{dom} f$ holds $f(i) \subseteq g(i)$.

Let us note that the predicate $f \dot{\subseteq} g$ is reflexive.

Let I, J be sets, let A be a many sorted set indexed by I, and let B be a many sorted set indexed by J. Let us observe that $A \dot{\subseteq} B$ if and only if:

(Def. 2) $I \subseteq J$ and for every set i such that $i \in I$ holds $A(i) \subseteq B(i)$.

Next we state three propositions:

(8)[1] Let I, J be sets, A be a many sorted set indexed by I, and B be a many sorted set indexed by J. If $A \dot{\subseteq} B$ and $B \dot{\subseteq} A$, then $A = B$.

(9) Let I, J, K be sets, A be a many sorted set indexed by I, B be a many sorted set indexed by J, and C be a many sorted set indexed by K. If $A \dot{\subseteq} B$ and $B \dot{\subseteq} C$, then $A \dot{\subseteq} C$.

(10) Let I be a set, A be a many sorted set indexed by I, and B be a many sorted set indexed by I. Then $A \dot{\subseteq} B$ if and only if $A \subseteq B$.

3. A bit of lambda calculus

In this article we present several logical schemes. The scheme *OnSingletons* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\langle o, \mathcal{F}(o)\rangle; o$ ranges over elements of $\mathcal{A} : \mathcal{P}[o]\}$ is a function

for all values of the parameters.

The scheme *DomOnSingletons* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\operatorname{dom} \mathcal{B} = \{o; o$ ranges over elements of $\mathcal{A} : \mathcal{P}[o]\}$

provided the following condition is met:

- $\mathcal{B} = \{\langle o, \mathcal{F}(o)\rangle; o$ ranges over elements of $\mathcal{A} : \mathcal{P}[o]\}$.

The scheme *ValOnSingletons* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{B}(\mathcal{C}) = \mathcal{F}(\mathcal{C})$

provided the parameters have the following properties:

- $\mathcal{B} = \{\langle o, \mathcal{F}(o)\rangle; o$ ranges over elements of $\mathcal{A} : \mathcal{P}[o]\}$, and
- $\mathcal{P}[\mathcal{C}]$.

4. More on old categories

The following propositions are true:

(11) For every category C and for all objects i, j, k of C holds $[: \hom(j,k), \hom(i,j) :] \subseteq \operatorname{dom}($the composition of $C)$.

(12) For every category C and for all objects i, j, k of C holds (the composition of $C)^\circ[: \hom(j,k), \hom(i,j) :] \subseteq \hom(i,k)$.

[1] The proposition (7) has been removed.

Let C be a category structure. The functor $\mathrm{HomSets}_C$ yielding a many sorted set indexed by $[:$ the objects of C, the objects of $C:]$ is defined as follows:

(Def. 3) For all objects i, j of C holds $\mathrm{HomSets}_C(i, j) = \hom(i, j)$.

We now state the proposition

(13) For every category C and for every object i of C holds $\mathrm{id}_i \in \mathrm{HomSets}_C(i, i)$.

Let C be a category. The functor $\mathrm{Composition}_C$ yields a binary composition of $\mathrm{HomSets}_C$ and is defined as follows:

(Def. 4) For all objects i, j, k of C holds $\mathrm{Composition}_C(i, j, k) =$ (the composition of C)$\restriction[:\mathrm{HomSets}_C(j, k), \mathrm{HomSets}_C(i, j):]$.

The following propositions are true:

(14) Let C be a category and i, j, k be objects of C. Suppose $\hom(i, j) \neq \emptyset$ and $\hom(j, k) \neq \emptyset$. Let f be a morphism from i to j and g be a morphism from j to k. Then $\mathrm{Composition}_C(i, j, k)(g, f) = g \cdot f$.

(15) For every category C holds $\mathrm{Composition}_C$ is associative.

(16) For every category C holds $\mathrm{Composition}_C$ has left units and right units.

5. Transforming an Old Category into a New One

Let C be a category. The functor $\mathrm{Alter}(C)$ yields a strict non empty category structure and is defined by:

(Def. 5) $\mathrm{Alter}(C) = \langle$the objects of C, $\mathrm{HomSets}_C, \mathrm{Composition}_C\rangle$.

The following three propositions are true:

(17) For every category C holds $\mathrm{Alter}(C)$ is associative.

(18) For every category C holds $\mathrm{Alter}(C)$ has units.

(19) For every category C holds $\mathrm{Alter}(C)$ is transitive.

Let C be a category. One can verify that $\mathrm{Alter}(C)$ is transitive and associative and has units.

6. More on New Categories

Let us observe that there exists a graph which is non empty and strict.

Let C be a graph. We say that C is reflexive if and only if:

(Def. 6) For every set x such that $x \in$ the carrier of C holds (the arrows of C)$(x, x) \neq \emptyset$.

Let C be a non empty graph. Let us observe that C is reflexive if and only if:

(Def. 7) For every object o of C holds $\langle o, o\rangle \neq \emptyset$.

Let C be a non empty transitive category structure. Let us observe that C is associative if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let o_1, o_2, o_3, o_4 be objects of C, f be a morphism from o_1 to o_2, g be a morphism from o_2 to o_3, and h be a morphism from o_3 to o_4. If $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$ and $\langle o_3, o_4\rangle \neq \emptyset$, then $(h \cdot g) \cdot f = h \cdot (g \cdot f)$.

Let C be a non empty category structure. Let us observe that C has units if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let o be an object of C. Then

(i) $\langle o,o\rangle \neq \emptyset$, and

(ii) there exists a morphism i from o to o such that for every object o' of C and for every morphism m' from o' to o and for every morphism m'' from o to o' holds if $\langle o',o\rangle \neq \emptyset$, then $i \cdot m' = m'$ and if $\langle o,o'\rangle \neq \emptyset$, then $m'' \cdot i = m''$.

Let us note that every non empty category structure which has units is also reflexive.
One can verify that there exists a graph which is non empty and reflexive.
Let us observe that there exists a category structure which is non empty and reflexive.

7. The empty category

The strict category structure $\emptyset_{CAT}$ is defined by:

(Def. 10) The carrier of $\emptyset_{CAT}$ is empty.

One can verify that $\emptyset_{CAT}$ is empty.
Let us observe that there exists a category structure which is empty and strict.
We now state the proposition

(20) For every empty strict category structure E holds $E = \emptyset_{CAT}$.

8. Subcategories

Let C be a category structure. A category structure is said to be a substructure of C if it satisfies the conditions (Def. 11).

(Def. 11)(i) The carrier of it $\subseteq$ the carrier of C,

(ii) the arrows of it $\dot{\subseteq}$ the arrows of C, and

(iii) the composition of it $\dot{\subseteq}$ the composition of C.

In the sequel C, C_1, C_2, C_3 denote category structures.
One can prove the following propositions:

(21) C is a substructure of C.

(22) If C_1 is a substructure of C_2 and C_2 is a substructure of C_3, then C_1 is a substructure of C_3.

(23) Let C_1, C_2 be category structures. Suppose C_1 is a substructure of C_2 and C_2 is a substructure of C_1. Then the category structure of C_1 = the category structure of C_2.

Let C be a category structure. Note that there exists a substructure of C which is strict.

Let C be a non empty category structure and let o be an object of C. The functor $\Box\restriction o$ yields a strict substructure of C and is defined by the conditions (Def. 12).

(Def. 12)(i) The carrier of $\Box\restriction o = \{o\}$,

(ii) the arrows of $\Box\restriction o = [\langle o,o\rangle \mapsto \langle o,o\rangle]$, and

(iii) the composition of $\Box\restriction o = \langle o, o, o\rangle \dot{\longmapsto}$(the composition of C)(o, o, o).

In the sequel C is a non empty category structure and o is an object of C.
Next we state the proposition

(24) For every object o' of $\Box\restriction o$ holds $o' = o$.

Let C be a non empty category structure and let o be an object of C. One can check that $\Box\restriction o$ is transitive and non empty.

Let C be a non empty category structure. Observe that there exists a substructure of C which is transitive, non empty, and strict.

The following proposition is true

(25) Let C be a transitive non empty category structure and D_1, D_2 be transitive non empty substructures of C. Suppose the carrier of $D_1 \subseteq$ the carrier of D_2 and the arrows of $D_1 \dot{\subseteq}$ the arrows of D_2. Then D_1 is a substructure of D_2.

Let C be a category structure and let D be a substructure of C. We say that D is full if and only if:

(Def. 13) The arrows of $D =$ (the arrows of C)$\restriction$[:the carrier of D, the carrier of D:].

Let C be a non empty category structure with units and let D be a substructure of C. We say that D is id-inheriting if and only if:

(Def. 14)(i) For every object o of D and for every object o' of C such that $o = o'$ holds $\mathrm{id}_{o'} \in \langle o, o \rangle$ if D is non empty,

(ii) TRUE, otherwise.

Let C be a category structure. Note that there exists a substructure of C which is full and strict.

Let C be a non empty category structure. Observe that there exists a substructure of C which is full, non empty, and strict.

Let C be a category and let o be an object of C. Observe that $\Box\restriction o$ is full and id-inheriting.

Let C be a category. Observe that there exists a substructure of C which is full, id-inheriting, non empty, and strict.

In the sequel C is a non empty transitive category structure.

The following propositions are true:

(26) Let D be a substructure of C. Suppose the carrier of $D =$ the carrier of C and the arrows of $D =$ the arrows of C. Then the category structure of $D =$ the category structure of C.

(27) Let D_1, D_2 be non empty transitive substructures of C. Suppose the carrier of $D_1 =$ the carrier of D_2 and the arrows of $D_1 =$ the arrows of D_2. Then the category structure of $D_1 =$ the category structure of D_2.

(28) Let D be a full substructure of C. Suppose the carrier of $D =$ the carrier of C. Then the category structure of $D =$ the category structure of C.

(29) Let C be a non empty category structure, D be a full non empty substructure of C, o_1, o_2 be objects of C, and p_1, p_2 be objects of D. If $o_1 = p_1$ and $o_2 = p_2$, then $\langle o_1, o_2 \rangle = \langle p_1, p_2 \rangle$.

(30) For every non empty category structure C and for every non empty substructure D of C holds every object of D is an object of C.

Let C be a transitive non empty category structure. Observe that every substructure of C which is full and non empty is also transitive.

Next we state three propositions:

(31) Let D_1, D_2 be full non empty substructures of C. Suppose the carrier of $D_1 =$ the carrier of D_2. Then the category structure of $D_1 =$ the category structure of D_2.

(32) Let C be a non empty category structure, D be a non empty substructure of C, o_1, o_2 be objects of C, and p_1, p_2 be objects of D. If $o_1 = p_1$ and $o_2 = p_2$, then $\langle p_1, p_2 \rangle \subseteq \langle o_1, o_2 \rangle$.

(33) Let C be a non empty transitive category structure, D be a non empty transitive substructure of C, and p_1, p_2, p_3 be objects of D. Suppose $\langle p_1, p_2 \rangle \neq \emptyset$ and $\langle p_2, p_3 \rangle \neq \emptyset$. Let o_1, o_2, o_3 be objects of C. Suppose $o_1 = p_1$ and $o_2 = p_2$ and $o_3 = p_3$. Let f be a morphism from o_1 to o_2, g be a morphism from o_2 to o_3, f_1 be a morphism from p_1 to p_2, and g_1 be a morphism from p_2 to p_3. If $f = f_1$ and $g = g_1$, then $g \cdot f = g_1 \cdot f_1$.

Let C be an associative transitive non empty category structure. Observe that every non empty substructure of C which is transitive is also associative.

One can prove the following proposition

(34) Let C be a non empty category structure, D be a non empty substructure of C, o_1, o_2 be objects of C, and p_1, p_2 be objects of D. If $o_1 = p_1$ and $o_2 = p_2$ and $\langle p_1, p_2 \rangle \neq \emptyset$, then every morphism from p_1 to p_2 is a morphism from o_1 to o_2.

Let C be a transitive non empty category structure with units. One can verify that every non empty substructure of C which is id-inheriting and transitive has also units.

Let C be a category. Observe that there exists a non empty substructure of C which is id-inheriting and transitive.

Let C be a category. A subcategory of C is an id-inheriting transitive substructure of C.

The following proposition is true

(35) Let C be a category, D be a non empty subcategory of C, o be an object of D, and o' be an object of C. If $o = o'$, then $\mathrm{id}_o = \mathrm{id}_{o'}$.

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