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Parallelity and Lines in Affine Spaces[1]

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Summary. In the article we introduce basic notions concerning affine spaces and investigate their fundamental properties. We define the function which to every nondegenerate pair of points assigns the line joining them and we extend the relation of parallelity to a relation between segments and lines, and between lines.

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The articles [3], [1], and [2] provide the notation and terminology for this paper.

We adopt the following convention: A_1 is an affine space and a, a', b, b', c, d, o, p, q, x, y, z, t, u, w are elements of A_1.

Let us consider A_1 and let us consider a, b, c. The predicate $\mathbf{L}(a,b,c)$ is defined as follows:

(Def. 1) $a,b \upharpoonleft\!\upharpoonright a,c$.

One can prove the following propositions:

(10)[1] For every a there exists b such that $a \neq b$.

(11) $x,y \upharpoonleft\!\upharpoonright y,x$ and $x,y \upharpoonleft\!\upharpoonright x,y$.

(12) $x,y \upharpoonleft\!\upharpoonright z,z$ and $z,z \upharpoonleft\!\upharpoonright x,y$.

(13) If $x,y \upharpoonleft\!\upharpoonright z,t$, then $x,y \upharpoonleft\!\upharpoonright t,z$ and $y,x \upharpoonleft\!\upharpoonright z,t$ and $y,x \upharpoonleft\!\upharpoonright t,z$ and $z,t \upharpoonleft\!\upharpoonright x,y$ and $z,t \upharpoonleft\!\upharpoonright y,x$ and $t,z \upharpoonleft\!\upharpoonright x,y$ and $t,z \upharpoonleft\!\upharpoonright y,x$.

(14) If $a \neq b$ and if $a,b \upharpoonleft\!\upharpoonright x,y$ and $a,b \upharpoonleft\!\upharpoonright z,t$ or $a,b \upharpoonleft\!\upharpoonright x,y$ and $z,t \upharpoonleft\!\upharpoonright a,b$ or $x,y \upharpoonleft\!\upharpoonright a,b$ and $z,t \upharpoonleft\!\upharpoonright a,b$ or $x,y \upharpoonleft\!\upharpoonright a,b$ and $a,b \upharpoonleft\!\upharpoonright z,t$, then $x,y \upharpoonleft\!\upharpoonright z,t$.

(15) If $\mathbf{L}(x,y,z)$, then $\mathbf{L}(x,z,y)$ and $\mathbf{L}(y,x,z)$ and $\mathbf{L}(y,z,x)$ and $\mathbf{L}(z,x,y)$ and $\mathbf{L}(z,y,x)$.

(16) $\mathbf{L}(x,x,y)$ and $\mathbf{L}(x,y,y)$ and $\mathbf{L}(x,y,x)$.

(17) If $x \neq y$ and $\mathbf{L}(x,y,z)$ and $\mathbf{L}(x,y,t)$ and $\mathbf{L}(x,y,u)$, then $\mathbf{L}(z,t,u)$.

(18) If $x \neq y$ and $\mathbf{L}(x,y,z)$ and $x,y \upharpoonleft\!\upharpoonright z,t$, then $\mathbf{L}(x,y,t)$.

(19) If $\mathbf{L}(x,y,z)$ and $\mathbf{L}(x,y,t)$, then $x,y \upharpoonleft\!\upharpoonright z,t$.

(20) If $u \neq z$ and $\mathbf{L}(x,y,u)$ and $\mathbf{L}(x,y,z)$ and $\mathbf{L}(u,z,w)$, then $\mathbf{L}(x,y,w)$.

(21) There exist x, y, z such that not $\mathbf{L}(x,y,z)$.

[1] Supported by RPBP.III-24.C2.
[1] The propositions (1)–(9) have been removed.

(22) If $x \neq y$, then there exists z such that not $\mathbf{L}(x,y,z)$.

(23) If not $\mathbf{L}(o,a,b)$ and $\mathbf{L}(o,b,b')$ and $a,b \upuparrows a,b'$, then $b = b'$.

Let us consider A_1, a, b. The functor $\mathrm{Line}(a,b)$ yields a subset of A_1 and is defined as follows:

(Def. 2) For every x holds $x \in \mathrm{Line}(a,b)$ iff $\mathbf{L}(a,b,x)$.

In the sequel A, C, D, K denote subsets of A_1.
Next we state the proposition

(25)[2] $\mathrm{Line}(a,b) = \mathrm{Line}(b,a)$.

Let us consider A_1, a, b. Let us notice that the functor $\mathrm{Line}(a,b)$ is commutative.
The following three propositions are true:

(26) $a \in \mathrm{Line}(a,b)$ and $b \in \mathrm{Line}(a,b)$.

(27) If $c \in \mathrm{Line}(a,b)$ and $d \in \mathrm{Line}(a,b)$ and $c \neq d$, then $\mathrm{Line}(c,d) \subseteq \mathrm{Line}(a,b)$.

(28) If $c \in \mathrm{Line}(a,b)$ and $d \in \mathrm{Line}(a,b)$ and $a \neq b$, then $\mathrm{Line}(a,b) \subseteq \mathrm{Line}(c,d)$.

Let us consider A_1 and let us consider A. We say that A is line if and only if:

(Def. 3) There exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a,b)$.

We introduce A is a line as a synonym of A is line.
One can prove the following propositions:

(30)[3] For all a, b, A, C such that A is a line and C is a line and $a \in A$ and $b \in A$ and $a \in C$ and $b \in C$ holds $a = b$ or $A = C$.

(31) If A is a line, then there exist a, b such that $a \in A$ and $b \in A$ and $a \neq b$.

(32) If A is a line, then there exists b such that $a \neq b$ and $b \in A$.

(33) $\mathbf{L}(a,b,c)$ iff there exists A such that A is a line and $a \in A$ and $b \in A$ and $c \in A$.

Let us consider A_1, let us consider a, b, and let us consider A. We say that a, b // A if and only if:

(Def. 4) There exist c, d such that $c \neq d$ and $A = \mathrm{Line}(c,d)$ and $a,b \upuparrows c,d$.

Let us consider A_1, A, C. We say that A // C if and only if:

(Def. 5) There exist a, b such that $A = \mathrm{Line}(a,b)$ and $a \neq b$ and a, b // C.

Next we state a number of propositions:

(36)[4] If $c \in \mathrm{Line}(a,b)$ and $a \neq b$, then $d \in \mathrm{Line}(a,b)$ iff $a,b \upuparrows c,d$.

(37) If A is a line and $a \in A$, then $b \in A$ iff a, b // A.

(38) $a \neq b$ and $A = \mathrm{Line}(a,b)$ iff A is a line and $a \in A$ and $b \in A$ and $a \neq b$.

(39) If A is a line and $a \in A$ and $b \in A$ and $a \neq b$ and $\mathbf{L}(a,b,x)$, then $x \in A$.

(40) If there exist a, b such that a, b // A, then A is a line.

(41) If $c \in A$ and $d \in A$ and A is a line and $c \neq d$, then a, b // A iff $a,b \upuparrows c,d$.

[2] The proposition (24) has been removed.
[3] The proposition (29) has been removed.
[4] The propositions (34) and (35) have been removed.

(43)[5] If $a \neq b$, then a, b // Line(a,b).

(44) If A is a line, then a, b // A iff there exist c, d such that $c \neq d$ and $c \in A$ and $d \in A$ and $a,b \upuparrows c,d$.

(45) If A is a line and a, b // A and c, d // A, then $a,b \upuparrows c,d$.

(46) If a, b // A and $a,b \upuparrows p,q$ and $a \neq b$, then p, q // A.

(47) If A is a line, then a, a // A.

(48) If a, b // A, then b, a // A.

(49) If a, b // A and $a \notin A$, then $b \notin A$.

(50) If A // C, then A is a line and C is a line.

(51) A // C iff there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $a,b \upuparrows c,d$ and $A = \mathrm{Line}(a,b)$ and $C = \mathrm{Line}(c,d)$.

(52) If A is a line and C is a line and $a \in A$ and $b \in A$ and $c \in C$ and $d \in C$ and $a \neq b$ and $c \neq d$, then A // C iff $a,b \upuparrows c,d$.

(53) If $a \in A$ and $b \in A$ and $c \in C$ and $d \in C$ and A // C, then $a,b \upuparrows c,d$.

(54) If $a \in A$ and $b \in A$ and A // C, then a, b // C.

(55) If A is a line, then A // A.

(56) If A // C, then C // A.

Let us consider A_1, A, C. Let us note that the predicate A // C is symmetric.
The following propositions are true:

(57) If a, b // A and A // C, then a, b // C.

(58) Suppose A // C and C // D or A // C and D // C or C // A and C // D or C // A and D // C. Then A // D.

(59) If A // C and $p \in A$ and $p \in C$, then $A = C$.

(60) If $x \in K$ and $a \notin K$ and a, b // K, then $a = b$ or not $\mathbf{L}(x,a,b)$.

(61) If a, b // K and a', b' // K and $\mathbf{L}(p,a,a')$ and $\mathbf{L}(p,b,b')$ and $p \in K$ and $a \notin K$ and $a = b$, then $a' = b'$.

(62) If A is a line and $a \in A$ and $b \in A$ and $c \in A$ and $a \neq b$ and $a,b \upuparrows c,d$, then $d \in A$.

(63) For all a, A such that A is a line there exists C such that $a \in C$ and A // C.

(64) If A // C and A // D and $p \in C$ and $p \in D$, then $C = D$.

(65) If A is a line and $a \in A$ and $b \in A$ and $c \in A$ and $d \in A$, then $a,b \upuparrows c,d$.

(66) If A is a line and $a \in A$ and $b \in A$, then a, b // A.

(67) If a, b // A and a, b // C and $a \neq b$, then A // C.

(68) If not $\mathbf{L}(o,a,b)$ and $\mathbf{L}(o,a,a')$ and $\mathbf{L}(o,b,b')$ and $a,b \upuparrows a',b'$ and $a' = b'$, then $a' = o$ and $b' = o$.

(69) If not $\mathbf{L}(o,a,b)$ and $\mathbf{L}(o,a,a')$ and $\mathbf{L}(o,b,b')$ and $a,b \upuparrows a',b'$ and $a' = o$, then $b' = o$.

[5] The proposition (42) has been removed.

(70) If not $\mathbf{L}(o,a,b)$ and $\mathbf{L}(o,a,a')$ and $\mathbf{L}(o,b,b')$ and $\mathbf{L}(o,b,x)$ and $a,b \upuparrows a',b'$ and $a,b \upuparrows a',x$, then $b' = x$.

(71) For all a, b, A such that A is a line and $a \in A$ and $b \in A$ and $a \neq b$ holds $A = \mathrm{Line}(a,b)$.

We adopt the following rules: A_2 denotes an affine plane, a, b, c, d, x, p denote elements of A_2, and A, C denote subsets of A_2.

Next we state three propositions:

(72) If A is a line and C is a line and A not // C, then there exists x such that $x \in A$ and $x \in C$.

(73) If A is a line and a, b not // A, then there exists x such that $x \in A$ and $\mathbf{L}(a,b,x)$.

(74) If $a,b \not\upuparrows c,d$, then there exists p such that $\mathbf{L}(a,b,p)$ and $\mathbf{L}(c,d,p)$.

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