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Definable Functions

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Summary. The article is continuation of [4] and [3]. It deals with concepts of variables occurring in a formula and free variables, replacing of variables in a formula and definable functions. The goal of it is to create a base of facts which are neccesary to show that every model of ZF set theory is a good model, i.e. it is closed with respect to fundamental settheoretical operations (union, intersection, Cartesian product etc.). The base includes the facts concerning with the composition and conditional sum of two definable functions.

MML Identifier: ZFMODEL2.

WWW: http://mizar.org/JFM/Vol2/zfmodel2.html

The articles [10], [9], [11], [12], [5], [8], [7], [6], [1], [2], [3], and [4] provide the notation and terminology for this paper.

For simplicity, we use the following convention: x, y, z, x_1, x_2, x_3, x_4 are variables, M is a non empty set, i, j are natural numbers, m, m_1, m_2, m_3, m_4 are elements of M, H, H_1, H_2 are ZF-formulae, and v, v_1, v_2 are functions from VAR into M.

The following propositions are true:

(1) $\mathrm{Free}(H(\frac{x}{y})) \subseteq (\mathrm{Free}\,H \setminus \{x\}) \cup \{y\}$.

(2) If $y \notin \mathrm{Var}_H$, then if $x \in \mathrm{Free}\,H$, then $\mathrm{Free}(H(\frac{x}{y})) = (\mathrm{Free}\,H \setminus \{x\}) \cup \{y\}$ and if $x \notin \mathrm{Free}\,H$, then $\mathrm{Free}(H(\frac{x}{y})) = \mathrm{Free}\,H$.

(3) Var_H is finite.

(4) There exists i such that for every j such that $\mathrm{x}_j \in \mathrm{Var}_H$ holds $j < i$ and there exists x such that $x \notin \mathrm{Var}_H$.

(5) If $x \notin \mathrm{Var}_H$, then $M, v \models H$ iff $M, v \models \forall_x H$.

(6) If $x \notin \mathrm{Var}_H$, then $M, v \models H$ iff $M, v(\frac{x}{m}) \models H$.

(7) If $x \neq y$ and $y \neq z$ and $z \neq x$, then $v(\frac{x}{m_1})(\frac{y}{m_2})(\frac{z}{m_3}) = v(\frac{z}{m_3})(\frac{y}{m_2})(\frac{x}{m_1})$ and $v(\frac{x}{m_1})(\frac{y}{m_2})(\frac{z}{m_3}) = v(\frac{y}{m_2})(\frac{z}{m_3})(\frac{x}{m_1})$.

(8) Suppose $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then $v(\frac{x_1}{m_1})(\frac{x_2}{m_2})(\frac{x_3}{m_3})(\frac{x_4}{m_4}) = v(\frac{x_2}{m_2})(\frac{x_3}{m_3})(\frac{x_4}{m_4})(\frac{x_1}{m_1})$ and $v(\frac{x_1}{m_1})(\frac{x_2}{m_2})(\frac{x_3}{m_3})(\frac{x_4}{m_4}) = v(\frac{x_3}{m_3})(\frac{x_4}{m_4})(\frac{x_1}{m_1})(\frac{x_2}{m_2})$ and $v(\frac{x_1}{m_1})(\frac{x_2}{m_2})(\frac{x_3}{m_3})(\frac{x_4}{m_4}) = v(\frac{x_4}{m_4})(\frac{x_2}{m_2})(\frac{x_3}{m_3})(\frac{x_1}{m_1})$.

(9) $v(\frac{x_1}{m_1})(\frac{x_2}{m_2})(\frac{x_1}{m}) = v(\frac{x_2}{m_2})(\frac{x_1}{m})$ and $v(\frac{x_1}{m_1})(\frac{x_2}{m_2})(\frac{x_3}{m_3})(\frac{x_1}{m}) = v(\frac{x_2}{m_2})(\frac{x_3}{m_3})(\frac{x_1}{m})$ and $v(\frac{x_1}{m_1})(\frac{x_2}{m_2})(\frac{x_3}{m_3})(\frac{x_4}{m_4})(\frac{x_1}{m}) = v(\frac{x_2}{m_2})(\frac{x_3}{m_3})(\frac{x_4}{m_4})(\frac{x_1}{m})$.

(10) If $x \notin \mathrm{Free}\,H$, then $M, v \models H$ iff $M, v(\frac{x}{m}) \models H$.

(11) If $x_0 \notin \mathrm{Free}\, H$ and $M, v \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H \Leftrightarrow x_4{=}(x_0))$, then for all m_1, m_2 holds $f'_H[v](m_1) = m_2$ iff $M, v(\frac{x_3}{m_1})(\frac{x_4}{m_2}) \models H$.

(12) If $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $M \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H \Leftrightarrow x_4{=}(x_0))$, then $f'_H[v] = f_H[M]$.

(13) If $x \notin \mathrm{Var}_H$, then $M, v \models H(\frac{y}{x})$ iff $M, v(\frac{y}{v(x)}) \models H$.

(14) If $x \notin \mathrm{Var}_H$ and $M, v \models H$, then $M, v(\frac{x}{v(y)}) \models H(\frac{y}{x})$.

(15) Suppose $x_0 \notin \mathrm{Free}\, H$ and $M, v \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H \Leftrightarrow x_4{=}(x_0))$ and $x \notin \mathrm{Var}_H$ and $y \neq x_3$ and $y \neq x_4$ and $y \notin \mathrm{Free}\, H$ and $x \neq x_0$ and $x \neq x_3$ and $x \neq x_4$. Then $x_0 \notin \mathrm{Free}(H(\frac{y}{x}))$ and $M, v(\frac{x}{v(y)}) \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H(\frac{y}{x}) \Leftrightarrow x_4{=}(x_0))$ and $f'_H[v] = f'_{H(\frac{y}{x})}[v(\frac{x}{v(y)})]$.

(16) If $x \notin \mathrm{Var}_H$, then $M \models H(\frac{y}{x})$ iff $M \models H$.

(17) Suppose $x_0 \notin \mathrm{Free}\, H_1$ and $M, v_1 \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H_1 \Leftrightarrow x_4{=}(x_0))$. Then there exist H_2, v_2 such that for every j such that $j < i$ and $x_j \in \mathrm{Var}_{(H_2)}$ holds $j = 3$ or $j = 4$ and $x_0 \notin \mathrm{Free}\, H_2$ and $M, v_2 \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H_2 \Leftrightarrow x_4{=}(x_0))$ and $f'_{H_1}[v_1] = f'_{H_2}[v_2]$.

(18) Suppose $x_0 \notin \mathrm{Free}\, H_1$ and $M, v_1 \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H_1 \Leftrightarrow x_4{=}(x_0))$. Then there exist H_2, v_2 such that $\mathrm{Free}\, H_1 \cap \mathrm{Free}\, H_2 \subseteq \{x_3, x_4\}$ and $x_0 \notin \mathrm{Free}\, H_2$ and $M, v_2 \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H_2 \Leftrightarrow x_4{=}(x_0))$ and $f'_{H_1}[v_1] = f'_{H_2}[v_2]$.

In the sequel F, G are functions.

The following propositions are true:

(19) If F is definable in M and G is definable in M, then $F \cdot G$ is definable in M.

(20) Suppose $x_0 \notin \mathrm{Free}\, H$. Then $M, v \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H \Leftrightarrow x_4{=}(x_0))$ if and only if for every m_1 there exists m_2 such that for every m_3 holds $M, v(\frac{x_3}{m_1})(\frac{x_4}{m_3}) \models H$ iff $m_3 = m_2$.

(21) Suppose F is definable in M and G is definable in M and $\mathrm{Free}\, H \subseteq \{x_3\}$. Let F_1 be a function. Suppose $\mathrm{dom}\, F_1 = M$ and for every v holds if $M, v \models H$, then $F_1(v(x_3)) = F(v(x_3))$ and if $M, v \models \neg H$, then $F_1(v(x_3)) = G(v(x_3))$. Then F_1 is definable in M.

(22) Suppose F is parametrically definable in M and G is parametrically definable in M. Then $G \cdot F$ is parametrically definable in M.

(23) Suppose that

(i) $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H_1$,

(ii) $M, v \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H_1 \Leftrightarrow x_4{=}(x_0))$,

(iii) $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H_2$,

(iv) $M, v \models \forall_{x_3} \exists_{x_0} \forall_{x_4} (H_2 \Leftrightarrow x_4{=}(x_0))$,

(v) $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$, and

(vi) $x_4 \notin \mathrm{Free}\, H$.

Let F_1 be a function. Suppose $\mathrm{dom}\, F_1 = M$ and for every m holds if $M, v(\frac{x_3}{m}) \models H$, then $F_1(m) = f'_{H_1}[v](m)$ and if $M, v(\frac{x_3}{m}) \models \neg H$, then $F_1(m) = f'_{H_2}[v](m)$. Then F_1 is parametrically definable in M.

(24) id_M is definable in M.

(25) id_M is parametrically definable in M.

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