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Some Basic Properties of Sets

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Summary. In this article some basic theorems about singletons, pairs, power sets, unions of families of sets, and the cartesian product of two sets are proved.

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The article [1] provides the notation and terminology for this paper.

1. Cartesian product of sets

In this paper x, x_1, x_2, y, y_1, y_2, z, A, B, X, X_1, X_2, X_3, X_4, Y, Y_1, Y_2, Z, N, M denote sets.

Let x, y be sets. One can verify that $\langle x, y \rangle$ is non empty.

Let us consider X. The functor 2^X is defined by:

(Def. 1) $Z \in 2^X$ iff $Z \subseteq X$.

Let us consider X_1, X_2. The functor $[: X_1, X_2 :]$ is defined as follows:

(Def. 2) $z \in [: X_1, X_2 :]$ iff there exist x, y such that $x \in X_1$ and $y \in X_2$ and $z = \langle x, y \rangle$.

Let us consider X_1, X_2, X_3. The functor $[: X_1, X_2, X_3 :]$ is defined by:

(Def. 3) $[: X_1, X_2, X_3 :] = [: [: X_1, X_2 :], X_3 :]$.

Let us consider X_1, X_2, X_3, X_4. The functor $[: X_1, X_2, X_3, X_4 :]$ is defined by:

(Def. 4) $[: X_1, X_2, X_3, X_4 :] = [: [: X_1, X_2, X_3 :], X_4 :]$.

2. Basic properties of sets

We now state a number of propositions:

(1) $2^{\emptyset} = \{\emptyset\}$.

(2) $\bigcup \emptyset = \emptyset$.

(6)[1] If $\{x\} \subseteq \{y\}$, then $x = y$.

(8)[2] If $\{x\} = \{y_1, y_2\}$, then $x = y_1$.

[1] The propositions (3)–(5) have been removed.
[2] The proposition (7) has been removed.

(9) If $\{x\} = \{y_1, y_2\}$, then $y_1 = y_2$.

(10) If $\{x_1, x_2\} = \{y_1, y_2\}$, then $x_1 = y_1$ or $x_1 = y_2$.

(12)[3] $\{x\} \subseteq \{x, y\}$.

(13) If $\{x\} \cup \{y\} = \{x\}$, then $x = y$.

(14) $\{x\} \cup \{x, y\} = \{x, y\}$.

(16)[4] If $\{x\}$ misses $\{y\}$, then $x \neq y$.

(17) If $x \neq y$, then $\{x\}$ misses $\{y\}$.

(18) If $\{x\} \cap \{y\} = \{x\}$, then $x = y$.

(19) $\{x\} \cap \{x, y\} = \{x\}$.

(20) $\{x\} \setminus \{y\} = \{x\}$ iff $x \neq y$.

(21) If $\{x\} \setminus \{y\} = \emptyset$, then $x = y$.

(22) $\{x\} \setminus \{x, y\} = \emptyset$.

(23) If $x \neq y$, then $\{x, y\} \setminus \{y\} = \{x\}$.

(24) If $\{x\} \subseteq \{y\}$, then $x = y$.

(25) If $\{z\} \subseteq \{x, y\}$, then $z = x$ or $z = y$.

(26) If $\{x, y\} \subseteq \{z\}$, then $x = z$.

(27) If $\{x, y\} \subseteq \{z\}$, then $\{x, y\} = \{z\}$.

(28) If $\{x_1, x_2\} \subseteq \{y_1, y_2\}$, then $x_1 = y_1$ or $x_1 = y_2$.

(29) If $x \neq y$, then $\{x\} \dot{-} \{y\} = \{x, y\}$.

(30) $2^{\{x\}} = \{\emptyset, \{x\}\}$.

(31) $\bigcup \{x\} = x$.

(32) $\bigcup \{\{x\}, \{y\}\} = \{x, y\}$.

(33) If $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$, then $x_1 = y_1$ and $x_2 = y_2$.

(34) $\langle x, y \rangle \in [: \{x_1\}, \{y_1\} :]$ iff $x = x_1$ and $y = y_1$.

(35) $[: \{x\}, \{y\} :] = \{\langle x, y \rangle\}$.

(36) $[: \{x\}, \{y, z\} :] = \{\langle x, y \rangle, \langle x, z \rangle\}$ and $[: \{x, y\}, \{z\} :] = \{\langle x, z \rangle, \langle y, z \rangle\}$.

(37) $\{x\} \subseteq X$ iff $x \in X$.

(38) $\{x_1, x_2\} \subseteq Z$ iff $x_1 \in Z$ and $x_2 \in Z$.

(39) $Y \subseteq \{x\}$ iff $Y = \emptyset$ or $Y = \{x\}$.

(40) If $Y \subseteq X$ and $x \notin Y$, then $Y \subseteq X \setminus \{x\}$.

(41) If $X \neq \{x\}$ and $X \neq \emptyset$, then there exists y such that $y \in X$ and $y \neq x$.

(42) $Z \subseteq \{x_1, x_2\}$ iff $Z = \emptyset$ or $Z = \{x_1\}$ or $Z = \{x_2\}$ or $Z = \{x_1, x_2\}$.

(43) If $\{z\} = X \cup Y$, then $X = \{z\}$ and $Y = \{z\}$ or $X = \emptyset$ and $Y = \{z\}$ or $X = \{z\}$ and $Y = \emptyset$.

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[3] The proposition (11) has been removed.

[4] The proposition (15) has been removed.

(44) If $\{z\} = X \cup Y$ and $X \neq Y$, then $X = \emptyset$ or $Y = \emptyset$.

(45) If $\{x\} \cup X \subseteq X$, then $x \in X$.

(46) If $x \in X$, then $\{x\} \cup X = X$.

(47) If $\{x,y\} \cup Z \subseteq Z$, then $x \in Z$.

(48) If $x \in Z$ and $y \in Z$, then $\{x,y\} \cup Z = Z$.

(49) $\{x\} \cup X \neq \emptyset$.

(50) $\{x,y\} \cup X \neq \emptyset$.

(51) If $X \cap \{x\} = \{x\}$, then $x \in X$.

(52) If $x \in X$, then $X \cap \{x\} = \{x\}$.

(53) If $x \in Z$ and $y \in Z$, then $\{x,y\} \cap Z = \{x,y\}$.

(54) If $\{x\}$ misses X, then $x \notin X$.

(55) If $\{x,y\}$ misses Z, then $x \notin Z$.

(56) If $x \notin X$, then $\{x\}$ misses X.

(57) If $x \notin Z$ and $y \notin Z$, then $\{x,y\}$ misses Z.

(58) $\{x\}$ misses X or $\{x\} \cap X = \{x\}$.

(59) If $\{x,y\} \cap X = \{x\}$, then $y \notin X$ or $x = y$.

(60) If $x \in X$ and if $y \notin X$ or $x = y$, then $\{x,y\} \cap X = \{x\}$.

(63)[5] If $\{x,y\} \cap X = \{x,y\}$, then $x \in X$.

(64) $z \in X \setminus \{x\}$ iff $z \in X$ and $z \neq x$.

(65) $X \setminus \{x\} = X$ iff $x \notin X$.

(66) If $X \setminus \{x\} = \emptyset$, then $X = \emptyset$ or $X = \{x\}$.

(67) $\{x\} \setminus X = \{x\}$ iff $x \notin X$.

(68) $\{x\} \setminus X = \emptyset$ iff $x \in X$.

(69) $\{x\} \setminus X = \emptyset$ or $\{x\} \setminus X = \{x\}$.

(70) $\{x,y\} \setminus X = \{x\}$ iff $x \notin X$ but $y \in X$ or $x = y$.

(72)[6] $\{x,y\} \setminus X = \{x,y\}$ iff $x \notin X$ and $y \notin X$.

(73) $\{x,y\} \setminus X = \emptyset$ iff $x \in X$ and $y \in X$.

(74) $\{x,y\} \setminus X = \emptyset$ or $\{x,y\} \setminus X = \{x\}$ or $\{x,y\} \setminus X = \{y\}$ or $\{x,y\} \setminus X = \{x,y\}$.

(75) $X \setminus \{x,y\} = \emptyset$ iff $X = \emptyset$ or $X = \{x\}$ or $X = \{y\}$ or $X = \{x,y\}$.

(79)[7] If $A \subseteq B$, then $2^A \subseteq 2^B$.

(80) $\{A\} \subseteq 2^A$.

(81) $2^A \cup 2^B \subseteq 2^{A \cup B}$.

[5] The propositions (61) and (62) have been removed.
[6] The proposition (71) has been removed.
[7] The propositions (76)–(78) have been removed.

(82) If $2^{A} \cup 2^{B} = 2^{A \cup B}$, then A and B are $\subseteq$-comparable.

(83) $2^{A \cap B} = 2^{A} \cap 2^{B}$.

(84) $2^{A \setminus B} \subseteq \{\emptyset\} \cup (2^{A} \setminus 2^{B})$.

(86)[8] $2^{A \setminus B} \cup 2^{B \setminus A} \subseteq 2^{A \dot{-} B}$.

(92)[9] If $X \in A$, then $X \subseteq \bigcup A$.

(93) $\bigcup\{X, Y\} = X \cup Y$.

(94) If for every X such that $X \in A$ holds $X \subseteq Z$, then $\bigcup A \subseteq Z$.

(95) If $A \subseteq B$, then $\bigcup A \subseteq \bigcup B$.

(96) $\bigcup(A \cup B) = \bigcup A \cup \bigcup B$.

(97) $\bigcup(A \cap B) \subseteq \bigcup A \cap \bigcup B$.

(98) If for every X such that $X \in A$ holds X misses B, then $\bigcup A$ misses B.

(99) $\bigcup(2^{A}) = A$.

(100) $A \subseteq 2^{\bigcup A}$.

(101) If for all X, Y such that $X \neq Y$ and $X \in A \cup B$ and $Y \in A \cup B$ holds X misses Y, then $\bigcup(A \cap B) = \bigcup A \cap \bigcup B$.

(102) If $z \in [: X, Y :]$, then there exist x, y such that $\langle x, y \rangle = z$.

(103) If $A \subseteq [: X, Y :]$ and $z \in A$, then there exist x, y such that $x \in X$ and $y \in Y$ and $z = \langle x, y \rangle$.

(104) If $z \in [: X_1, Y_1 :] \cap [: X_2, Y_2 :]$, then there exist x, y such that $z = \langle x, y \rangle$ and $x \in X_1 \cap X_2$ and $y \in Y_1 \cap Y_2$.

(105) $[: X, Y :] \subseteq 2^{2^{X \cup Y}}$.

(106) $\langle x, y \rangle \in [: X, Y :]$ iff $x \in X$ and $y \in Y$.

(107) If $\langle x, y \rangle \in [: X, Y :]$, then $\langle y, x \rangle \in [: Y, X :]$.

(108) If for all x, y holds $\langle x, y \rangle \in [: X_1, Y_1 :]$ iff $\langle x, y \rangle \in [: X_2, Y_2 :]$, then $[: X_1, Y_1 :] = [: X_2, Y_2 :]$.

(109) If $A \subseteq [: X, Y :]$ and for all x, y such that $\langle x, y \rangle \in A$ holds $\langle x, y \rangle \in B$, then $A \subseteq B$.

(110) If $A \subseteq [: X_1, Y_1 :]$ and $B \subseteq [: X_2, Y_2 :]$ and for all x, y holds $\langle x, y \rangle \in A$ iff $\langle x, y \rangle \in B$, then $A = B$.

(111) If for every z such that $z \in A$ there exist x, y such that $z = \langle x, y \rangle$ and for all x, y such that $\langle x, y \rangle \in A$ holds $\langle x, y \rangle \in B$, then $A \subseteq B$.

(112) Suppose that

(i) for every z such that $z \in A$ there exist x, y such that $z = \langle x, y \rangle$,

(ii) for every z such that $z \in B$ there exist x, y such that $z = \langle x, y \rangle$, and

(iii) for all x, y holds $\langle x, y \rangle \in A$ iff $\langle x, y \rangle \in B$.

Then $A = B$.

(113) $[: X, Y :] = \emptyset$ iff $X = \emptyset$ or $Y = \emptyset$.

(114) If $X \neq \emptyset$ and $Y \neq \emptyset$ and $[: X, Y :] = [: Y, X :]$, then $X = Y$.

(115) If $[: X, X :] = [: Y, Y :]$, then $X = Y$.

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[8] The proposition (85) has been removed.

[9] The propositions (87)–(91) have been removed.

(116) If $X \subseteq [: X, X :]$, then $X = \emptyset$.

(117) If $Z \neq \emptyset$ and if $[: X, Z :] \subseteq [: Y, Z :]$ or $[: Z, X :] \subseteq [: Z, Y :]$, then $X \subseteq Y$.

(118) If $X \subseteq Y$, then $[: X, Z :] \subseteq [: Y, Z :]$ and $[: Z, X :] \subseteq [: Z, Y :]$.

(119) If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then $[: X_1, X_2 :] \subseteq [: Y_1, Y_2 :]$.

(120) $[: X \cup Y, Z :] = [: X, Z :] \cup [: Y, Z :]$ and $[: Z, X \cup Y :] = [: Z, X :] \cup [: Z, Y :]$.

(121) $[: X_1 \cup X_2, Y_1 \cup Y_2 :] = [: X_1, Y_1 :] \cup [: X_1, Y_2 :] \cup [: X_2, Y_1 :] \cup [: X_2, Y_2 :]$.

(122) $[: X \cap Y, Z :] = [: X, Z :] \cap [: Y, Z :]$ and $[: Z, X \cap Y :] = [: Z, X :] \cap [: Z, Y :]$.

(123) $[: X_1 \cap X_2, Y_1 \cap Y_2 :] = [: X_1, Y_1 :] \cap [: X_2, Y_2 :]$.

(124) If $A \subseteq X$ and $B \subseteq Y$, then $[: A, Y :] \cap [: X, B :] = [: A, B :]$.

(125) $[: X \setminus Y, Z :] = [: X, Z :] \setminus [: Y, Z :]$ and $[: Z, X \setminus Y :] = [: Z, X :] \setminus [: Z, Y :]$.

(126) $[: X_1, X_2 :] \setminus [: Y_1, Y_2 :] = [: X_1 \setminus Y_1, X_2 :] \cup [: X_1, X_2 \setminus Y_2 :]$.

(127) If X_1 misses X_2 or Y_1 misses Y_2, then $[: X_1, Y_1 :]$ misses $[: X_2, Y_2 :]$.

(128) $\langle x, y \rangle \in [: \{z\}, Y :]$ iff $x = z$ and $y \in Y$.

(129) $\langle x, y \rangle \in [: X, \{z\} :]$ iff $x \in X$ and $y = z$.

(130) If $X \neq \emptyset$, then $[: \{x\}, X :] \neq \emptyset$ and $[: X, \{x\} :] \neq \emptyset$.

(131) If $x \neq y$, then $[: \{x\}, X :]$ misses $[: \{y\}, Y :]$ and $[: X, \{x\} :]$ misses $[: Y, \{y\} :]$.

(132) $[: \{x, y\}, X :] = [: \{x\}, X :] \cup [: \{y\}, X :]$ and $[: X, \{x, y\} :] = [: X, \{x\} :] \cup [: X, \{y\} :]$.

(134)[10] If $X_1 \neq \emptyset$ and $Y_1 \neq \emptyset$ and $[: X_1, Y_1 :] = [: X_2, Y_2 :]$, then $X_1 = X_2$ and $Y_1 = Y_2$.

(135) If $X \subseteq [: X, Y :]$ or $X \subseteq [: Y, X :]$, then $X = \emptyset$.

(136) There exists M such that

(i) $N \in M$,

(ii) for all X, Y such that $X \in M$ and $Y \subseteq X$ holds $Y \in M$,

(iii) for every X such that $X \in M$ holds $2^X \in M$, and

(iv) for every X such that $X \subseteq M$ holds $X \approx M$ or $X \in M$.

References

[1] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

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[10] The proposition (133) has been removed.