JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Reflection Theorem

Grzegorz Bancerek
Warsaw University
Białystok

Summary. The goal is show that the reflection theorem holds. The theorem is as usual in the Morse-Kelley theory of classes (MK). That theory works with universal class which consists of all sets and every class is a subclass of it. In this paper (and in another Mizar articles) we work in Tarski-Grothendieck (TG) theory (see [15]) which ensures the existence of sets that have properties like universal class (i.e. this theory is stronger than MK). The sets are introduced in [13] and some concepts of MK are modeled. The concepts are: the class *On* of all ordinal numbers belonging to the universe, subclasses, transfinite sequences of non-empty elements of universe, etc. The reflection theorem states that if A_ξ is an increasing and continuous transfinite sequence of non-empty sets and class $A = \bigcup_{\xi \in On} A_\xi$, then for every formula H there is a strictly increasing continuous mapping $F : On \to On$ such that if $\varkappa$ is a critical number of F (i.e. $F(\varkappa) = \varkappa > 0$) and $f \in A_\varkappa^{\mathbf{VAR}}$, then $A, f \models H \equiv A_\varkappa, f \models H$. The proof is based on [11]. Besides, in the article it is shown that every universal class is a model of ZF set theory if ω (the first infinite ordinal number) belongs to it. Some propositions concerning ordinal numbers and sequences of them are also present.

MML Identifier: ZF_REFLE.

WWW: http://mizar.org/JFM/Vol2/zf_refle.html

The articles [15], [14], [17], [16], [18], [9], [10], [3], [4], [5], [1], [12], [8], [13], [2], [6], and [7] provide the notation and terminology for this paper.

In this paper W is a universal class, H is a ZF-formula, and x, X are sets.

We now state several propositions:

(2)[1] $W \models$ the axiom of pairs.

(3) $W \models$ the axiom of unions.

(4) If $\omega \in W$, then $W \models$ the axiom of infinity.

(5) $W \models$ the axiom of power sets.

(6) For every H such that $\{x_0, x_1, x_2\}$ misses Free H holds $W \models$ the axiom of substitution for H.

(7) If $\omega \in W$, then W is a model of ZF.

For simplicity, we adopt the following rules: F is a function, A, B, C are ordinal numbers, a, b are ordinals of W, p_1 is a transfinite sequence of ordinals of W, and H is a ZF-formula.

Let us consider A, B. Let us observe that $A \subseteq B$ if and only if:

(Def. 1) For every C such that $C \in A$ holds $C \in B$.

[1] The proposition (1) has been removed.

In this article we present several logical schemes. The scheme *ALFA* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists F such that $\operatorname{dom} F = \mathcal{A}$ and for every element d of $\mathcal{A}$ there exists A such that $A = F(d)$ and $\mathcal{P}[d,A]$ and for every B such that $\mathcal{P}[d,B]$ holds $A \subseteq B$

provided the parameters satisfy the following condition:

- For every element d of $\mathcal{A}$ there exists A such that $\mathcal{P}[d,A]$.

The scheme *ALFA'Universe* deals with a universal class $\mathcal{A}$, a non empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists F such that
(i) $\operatorname{dom} F = \mathcal{B}$, and
(ii) for every element d of $\mathcal{B}$ there exists an ordinal a of $\mathcal{A}$ such that $a = F(d)$ and $\mathcal{P}[d,a]$ and for every ordinal b of $\mathcal{A}$ such that $\mathcal{P}[d,b]$ holds $a \subseteq b$

provided the following condition is satisfied:

- For every element d of $\mathcal{B}$ there exists an ordinal a of $\mathcal{A}$ such that $\mathcal{P}[d,a]$.

Next we state the proposition

(8) x is an ordinal of W iff $x \in \operatorname{On} W$.

In the sequel p_2 is a sequence of ordinal numbers.

Now we present three schemes. The scheme *OrdSeqOfUnivEx* deals with a universal class $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a transfinite sequence p_1 of ordinals of $\mathcal{A}$ such that for every ordinal a of $\mathcal{A}$ holds $\mathcal{P}[a, p_1(a)]$

provided the following conditions are met:

- For all ordinals a, b_1, b_2 of $\mathcal{A}$ such that $\mathcal{P}[a,b_1]$ and $\mathcal{P}[a,b_2]$ holds $b_1 = b_2$, and
- For every ordinal a of $\mathcal{A}$ there exists an ordinal b of $\mathcal{A}$ such that $\mathcal{P}[a,b]$.

The scheme *UOS Exist* deals with a universal class $\mathcal{A}$, an ordinal $\mathcal{B}$ of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an ordinal of $\mathcal{A}$, and a binary functor $\mathcal{G}$ yielding an ordinal of $\mathcal{A}$, and states that:

There exists a transfinite sequence p_1 of ordinals of $\mathcal{A}$ such that
(i) $p_1(\mathbf{0}_{\mathcal{A}}) = \mathcal{B}$,
(ii) for every ordinal a of $\mathcal{A}$ holds $p_1(\operatorname{succ} a) = \mathcal{F}(a, p_1(a))$, and
(iii) for every ordinal a of $\mathcal{A}$ such that $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number holds $p_1(a) = \mathcal{G}(a, p_1 \restriction a)$

for all values of the parameters.

The scheme *Universe Ind* deals with a universal class $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every ordinal a of $\mathcal{A}$ holds $\mathcal{P}[a]$

provided the following requirements are met:

- $\mathcal{P}[\mathbf{0}_{\mathcal{A}}]$,
- For every ordinal a of $\mathcal{A}$ such that $\mathcal{P}[a]$ holds $\mathcal{P}[\operatorname{succ} a]$, and
- Let a be an ordinal of $\mathcal{A}$. Suppose $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number and for every ordinal b of $\mathcal{A}$ such that $b \in a$ holds $\mathcal{P}[b]$. Then $\mathcal{P}[a]$.

Let f be a function, let W be a universal class, and let a be an ordinal of W. The functor $\bigcup_a f$ yielding a set is defined as follows:

(Def. 2) $\bigcup_a f = \bigcup (W \restriction (f \restriction \mathbf{R}_a))$.

Next we state several propositions:

(10)[2] For every transfinite sequence L and for every A holds $L \restriction \mathbf{R}_A$ is a transfinite sequence.

(11) For every sequence L of ordinal numbers and for every A holds $L \restriction \mathbf{R}_A$ is a sequence of ordinal numbers.

(12) $\bigcup p_2$ is an ordinal number.

(13) $\bigcup (X \restriction p_2)$ is an ordinal number.

[2] The proposition (9) has been removed.

(14) $\mathrm{On}(\mathbf{R}_A) = A$.

(15) $p_2 \restriction \mathbf{R}_A = p_2 \restriction A$.

Let p_1 be a sequence of ordinal numbers, let W be a universal class, and let a be an ordinal of W. Then $\bigcup_a p_1$ is an ordinal of W.

The following proposition is true

(17)[3] For every transfinite sequence p_1 of ordinals of W holds $\bigcup_a p_1 = \bigcup(p_1 \restriction a)$ and $\bigcup_a (p_1 \restriction a) = \bigcup(p_1 \restriction a)$.

Let W be a universal class and let a, b be ordinals of W. Then $a \cup b$ is an ordinal of W.

Let us consider W. Observe that there exists an element of W which is non empty.

Let us consider W. A subclass of W is a non empty subset of W.

Let us consider W and let I_1 be a transfinite sequence of elements of W. We say that I_1 is non empty set yielding if and only if:

(Def. 5)[4] $\mathrm{dom}\, I_1 = \mathrm{On}\, W$.

Let us consider W. Observe that there exists a transfinite sequence of elements of W which is non empty set yielding and non-empty.

Let us consider W. A transfinite sequence of non empty sets from W is a non-empty non empty set yielding transfinite sequence of elements of W.

Let us consider W and let L be a transfinite sequence of non empty sets from W. Then $\bigcup L$ is a subclass of W. Let us consider a. Then $L(a)$ is a non empty element of W.

In the sequel L is a transfinite sequence of non empty sets from W and f is a function from VAR into $L(a)$.

Next we state several propositions:

(23)[5] $a \in \mathrm{dom}\, L$.

(24) $L(a) \subseteq \bigcup L$.

(25) $\mathbb{N} \approx \mathrm{VAR}$.

(27)[6] $\sup X \subseteq \mathrm{succ} \bigcup \mathrm{On}\, X$.

(28) If $X \in W$, then $\sup X \in W$.

(29) Suppose that

(i) $\omega \in W$,

(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$, and

(iii) for every a such that $a \neq \emptyset$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$.

Let given H. Then there exists p_1 such that

(iv) p_1 is increasing and continuous, and

(v) for every a such that $p_1(a) = a$ and $\emptyset \neq a$ and for every f holds $\bigcup L, (\bigcup L)[f] \models H$ iff $L(a), f \models H$.

[3] The proposition (16) has been removed.
[4] The definitions (Def. 3) and (Def. 4) have been removed.
[5] The propositions (18)–(22) have been removed.
[6] The proposition (26) has been removed.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. A model of ZF set theory language. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zf_lang.html`.

[3] Grzegorz Bancerek. Models and satisfiability. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zf_model.html`.

[4] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[5] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal2.html`.

[6] Grzegorz Bancerek. Increasing and continuous ordinal sequences. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/ordinal4.html`.

[7] Grzegorz Bancerek. Replacing of variables in formulas of ZF theory. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/zf_lang1.html`.

[8] Grzegorz Bancerek. Tarski's classes and ranks. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/classes1.html`.

[9] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[10] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[11] Andrzej Mostowski. *Constructible Sets with Applications*. North Holland, 1969.

[12] Andrzej Nędzusiak. σ-fields and probability. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/prob_1.html`.

[13] Bogdan Nowak and Grzegorz Bancerek. Universal classes. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/classes2.html`.

[14] Andrzej Trybulec. Enumerated sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/enumset1.html`.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[16] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[17] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[18] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received August 10, 1990

Published January 2, 2004