JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

Duality in Relation Structures[1]

Grzegorz Bancerek
Warsaw University
Białystok

MML Identifier: YELLOW_7.

WWW: http://mizar.org/JFM/Vol8/yellow_7.html

The articles [12], [15], [16], [18], [17], [7], [8], [10], [1], [2], [6], [11], [13], [14], [3], [4], [20], [9], [5], and [19] provide the notation and terminology for this paper.

Let L be a relational structure. We introduce $L^{\rm op}$ as a synonym of $L^{\smile}$.

We now state several propositions:

(1) For every relational structure L and for all elements x, y of $L^{\rm op}$ holds $x \leq y$ iff $\curvearrowleft x \geq \curvearrowleft y$.

(2) Let L be a relational structure, x be an element of L, and y be an element of $L^{\rm op}$. Then

(i) $x \leq \curvearrowleft y$ iff $x^{\smile} \geq y$, and

(ii) $x \geq \curvearrowleft y$ iff $x^{\smile} \leq y$.

(3) For every relational structure L holds L is empty iff $L^{\rm op}$ is empty.

(4) For every relational structure L holds L is reflexive iff $L^{\rm op}$ is reflexive.

(5) For every relational structure L holds L is antisymmetric iff $L^{\rm op}$ is antisymmetric.

(6) For every relational structure L holds L is transitive iff $L^{\rm op}$ is transitive.

(7) For every non empty relational structure L holds L is connected iff $L^{\rm op}$ is connected.

Let L be a reflexive relational structure. Observe that $L^{\rm op}$ is reflexive.

Let L be a transitive relational structure. One can verify that $L^{\rm op}$ is transitive.

Let L be an antisymmetric relational structure. One can check that $L^{\rm op}$ is antisymmetric.

Let L be a connected non empty relational structure. One can check that $L^{\rm op}$ is connected.

One can prove the following propositions:

(8) Let L be a relational structure, x be an element of L, and X be a set. Then

(i) $x \leq X$ iff $x^{\smile} \geq X$, and

(ii) $x \geq X$ iff $x^{\smile} \leq X$.

(9) Let L be a relational structure, x be an element of $L^{\rm op}$, and X be a set. Then

(i) $x \leq X$ iff $\curvearrowleft x \geq X$, and

(ii) $x \geq X$ iff $\curvearrowleft x \leq X$.

[1] This work has been partially supported by the Office of Naval Research Grant N00014-95-1-1336.

(10) Let L be a relational structure and X be a set. Then sup X exists in L if and only if inf X exists in $L^{\rm op}$.

(11) Let L be a relational structure and X be a set. Then sup X exists in $L^{\rm op}$ if and only if inf X exists in L.

(12) Let L be a non empty relational structure and X be a set. If sup X exists in L or inf X exists in $L^{\rm op}$, then $\bigsqcup_L X = \bigsqcap_{(L^{\rm op})} X$.

(13) Let L be a non empty relational structure and X be a set. If inf X exists in L or sup X exists in $L^{\rm op}$, then $\bigsqcap_L X = \bigsqcup_{(L^{\rm op})} X$.

(14) Let L_1, L_2 be relational structures such that the relational structure of $L_1 =$ the relational structure of L_2 and L_1 has g.l.b.'s. Then L_2 has g.l.b.'s.

(15) Let L_1, L_2 be relational structures such that the relational structure of $L_1 =$ the relational structure of L_2 and L_1 has l.u.b.'s. Then L_2 has l.u.b.'s.

(16) For every relational structure L holds L has g.l.b.'s iff $L^{\rm op}$ has l.u.b.'s.

(17) For every non empty relational structure L holds L is complete iff $L^{\rm op}$ is complete.

Let L be a relational structure with g.l.b.'s. Observe that $L^{\rm op}$ has l.u.b.'s.
Let L be a relational structure with l.u.b.'s. Note that $L^{\rm op}$ has g.l.b.'s.
Let L be a complete non empty relational structure. One can check that $L^{\rm op}$ is complete.
One can prove the following propositions:

(18) Let L be a non empty relational structure, X be a subset of L, and Y be a subset of $L^{\rm op}$. If $X = Y$, then ${\rm fininfs}(X) = {\rm finsups}(Y)$ and ${\rm finsups}(X) = {\rm fininfs}(Y)$.

(19) Let L be a relational structure, X be a subset of L, and Y be a subset of $L^{\rm op}$. If $X = Y$, then $\downarrow X = \uparrow Y$ and $\uparrow X = \downarrow Y$.

(20) Let L be a non empty relational structure, x be an element of L, and y be an element of $L^{\rm op}$. If $x = y$, then $\downarrow x = \uparrow y$ and $\uparrow x = \downarrow y$.

(21) For every poset L with g.l.b.'s and for all elements x, y of L holds $x \sqcap y = x^{\smile} \sqcup y^{\smile}$.

(22) For every poset L with g.l.b.'s and for all elements x, y of $L^{\rm op}$ holds $\frown x \sqcap \frown y = x \sqcup y$.

(23) For every poset L with l.u.b.'s and for all elements x, y of L holds $x \sqcup y = x^{\smile} \sqcap y^{\smile}$.

(24) For every poset L with l.u.b.'s and for all elements x, y of $L^{\rm op}$ holds $\frown x \sqcup \frown y = x \sqcap y$.

(25) For every lattice L holds L is distributive iff $L^{\rm op}$ is distributive.

Let L be a distributive lattice. Observe that $L^{\rm op}$ is distributive.
The following propositions are true:

(26) Let L be a relational structure and x be a set. Then x is a directed subset of L if and only if x is a filtered subset of $L^{\rm op}$.

(27) Let L be a relational structure and x be a set. Then x is a directed subset of $L^{\rm op}$ if and only if x is a filtered subset of L.

(28) Let L be a relational structure and x be a set. Then x is a lower subset of L if and only if x is an upper subset of $L^{\rm op}$.

(29) Let L be a relational structure and x be a set. Then x is a lower subset of $L^{\rm op}$ if and only if x is an upper subset of L.

(30) For every relational structure L holds L is lower-bounded iff $L^{\rm op}$ is upper-bounded.

(31) For every relational structure L holds $L^{\rm op}$ is lower-bounded iff L is upper-bounded.

(32) For every relational structure L holds L is bounded iff $L^{\rm op}$ is bounded.

(33) For every lower-bounded antisymmetric non empty relational structure L holds $(\bot_L)^\smile = \top_{L^{\rm op}}$ and $\frown(\top_{L^{\rm op}}) = \bot_L$.

(34) For every upper-bounded antisymmetric non empty relational structure L holds $(\top_L)^\smile = \bot_{L^{\rm op}}$ and $\frown(\bot_{L^{\rm op}}) = \top_L$.

(35) Let L be a bounded lattice and x, y be elements of L. Then y is a complement of x if and only if $y^\smile$ is a complement of $x^\smile$.

(36) For every bounded lattice L holds L is complemented iff $L^{\rm op}$ is complemented.

Let L be a lower-bounded relational structure. Observe that $L^{\rm op}$ is upper-bounded.
Let L be an upper-bounded relational structure. Observe that $L^{\rm op}$ is lower-bounded.
Let L be a complemented bounded lattice. Note that $L^{\rm op}$ is complemented.
The following proposition is true

(37) For every Boolean lattice L and for every element x of L holds $\neg(x^\smile) = \neg x$.

Let L be a non empty relational structure. The functor $\neg_L$ yielding a map from L into $L^{\rm op}$ is defined as follows:

(Def. 1) For every element x of L holds $\neg_L(x) = \neg x$.

Let L be a Boolean lattice. One can verify that $\neg_L$ is one-to-one.
Let L be a Boolean lattice. One can check that $\neg_L$ is isomorphic.
One can prove the following propositions:

(38) For every Boolean lattice L holds L and $L^{\rm op}$ are isomorphic.

(39) Let S, T be non empty relational structures and f be a set. Then

(i) f is a map from S into T iff f is a map from $S^{\rm op}$ into T,

(ii) f is a map from S into T iff f is a map from S into $T^{\rm op}$, and

(iii) f is a map from S into T iff f is a map from $S^{\rm op}$ into $T^{\rm op}$.

(40) Let S, T be non empty relational structures, f be a map from S into T, and g be a map from S into $T^{\rm op}$ such that $f = g$. Then

(i) f is monotone iff g is antitone, and

(ii) f is antitone iff g is monotone.

(41) Let S, T be non empty relational structures, f be a map from S into $T^{\rm op}$, and g be a map from $S^{\rm op}$ into T such that $f = g$. Then

(i) f is monotone iff g is monotone, and

(ii) f is antitone iff g is antitone.

(42) Let S, T be non empty relational structures, f be a map from S into T, and g be a map from $S^{\rm op}$ into $T^{\rm op}$ such that $f = g$. Then

(i) f is monotone iff g is monotone, and

(ii) f is antitone iff g is antitone.

(43) Let S, T be non empty relational structures and f be a set. Then

(i) f is a connection between S and T iff f is a connection between $S^\smile$ and T,

(ii) f is a connection between S and T iff f is a connection between S and $T^\smile$, and

(iii) f is a connection between S and T iff f is a connection between $S^\smile$ and $T^\smile$.

(44) Let S, T be non empty posets, f_1 be a map from S into T, g_1 be a map from T into S, f_2 be a map from $S^\smile$ into $T^\smile$, and g_2 be a map from $T^\smile$ into $S^\smile$. If $f_1 = f_2$ and $g_1 = g_2$, then $\langle f_1, g_1\rangle$ is Galois iff $\langle g_2, f_2\rangle$ is Galois.

(45) Let J be a set, D be a non empty set, K be a many sorted set indexed by J, and F be a set of elements of D double indexed by K. Then $\mathrm{dom}_\kappa F(\kappa) = K$.

Let J, D be non empty sets, let K be a non-empty many sorted set indexed by J, let F be a set of elements of D double indexed by K, let j be an element of J, and let k be an element of $K(j)$. Then $F(j)(k)$ is an element of D.

We now state several propositions:

(46) Let L be a non empty relational structure, J be a set, K be a many sorted set indexed by J, and x be a set. Then x is a set of elements of L double indexed by K if and only if x is a set of elements of L^{op} double indexed by K.

(47) Let L be a complete lattice, J be a non empty set, K be a non-empty many sorted set indexed by J, and F be a set of elements of L double indexed by K. Then $\mathrm{Sup}(\mathrm{Infs}(F)) \leq \mathrm{Inf}(\mathrm{Sups}(\mathrm{Frege}(F)))$.

(48) Let L be a complete lattice. Then L is completely-distributive if and only if for every non empty set J and for every non-empty many sorted set K indexed by J and for every set F of elements of L double indexed by K holds $\mathrm{Sup}(\mathrm{Infs}(F)) = \mathrm{Inf}(\mathrm{Sups}(\mathrm{Frege}(F)))$.

(49) Let L be a complete antisymmetric non empty relational structure and F be a function. Then $\bigsqcup_L F = \bigsqcap_{(L^{\mathrm{op}})} F$ and $\bigsqcap_L F = \bigsqcup_{(L^{\mathrm{op}})} F$.

(50) Let L be a complete antisymmetric non empty relational structure and F be a function yielding function. Then $\underline{\bigsqcup}_L F = \overline{\bigsqcap}_{(L^{\mathrm{op}})} F$ and $\overline{\bigsqcap}_L F = \underline{\bigsqcup}_{(L^{\mathrm{op}})} F$.

Let us observe that every non empty relational structure which is completely-distributive is also complete.

One can check that there exists a non empty poset which is completely-distributive, trivial, and strict.

The following proposition is true

(51) For every non empty poset L holds L is completely-distributive iff L^{op} is completely-distributive.

References

[1] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[2] Grzegorz Bancerek. Cartesian product of functions. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/funct_6.html`.

[3] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/lattice3.html`.

[4] Grzegorz Bancerek. Bounds in posets and relational substructures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_0.html`.

[5] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_0.html`.

[6] Józef Białas. Group and field definitions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/realset1.html`.

[7] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[8] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[9] Czesław Byliński. Galois connections. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_1.html`.

[10] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[11] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[13] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[14] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/orders_1.html.

[15] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[16] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[17] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

[18] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_2.html.

[19] Mariusz Żynel. The equational characterization of continuous lattices. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/waybel_5.html.

[20] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/yellow_2.html.

Received November 12, 1996

Published January 2, 2004