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Properties of Relational Structures, Posets, Lattices and Maps[1]

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Summary. In the paper we present some auxiliary facts concerning posets and maps between them. Our main purpose, however is to give an account on complete lattices and lattices of ideals. A sufficient condition that a lattice might be complete, the fixed-point theorem and two remarks upon images of complete lattices in monotone maps, introduced in [9, pp. 8–9], can be found in Section 7. Section 8 deals with lattices of ideals. We examine the meet and join of two ideals. In order to show that the lattice of ideals is complete, the infinite intersection of ideals is investigated.

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The articles [15], [8], [17], [18], [6], [7], [13], [2], [1], [16], [14], [3], [10], [4], [11], [5], and [12] provide the notation and terminology for this paper.

1. Basic Facts

In this paper x, X, Y are sets.

The scheme *RelStrSubset* deals with a non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\{x; x$ ranges over elements of $\mathcal{A} : \mathcal{P}[x]\}$ is a subset of $\mathcal{A}$

for all values of the parameters.

Next we state four propositions:

(1) Let L be a non empty relational structure, x be an element of L, and X be a subset of L. Then $X \subseteq \downarrow x$ if and only if $X \leq x$.

(2) Let L be a non empty relational structure, x be an element of L, and X be a subset of L. Then $X \subseteq \uparrow x$ if and only if $x \leq X$.

(3) Let L be an antisymmetric transitive relational structure with l.u.b.'s and X, Y be sets. Suppose sup X exists in L and sup Y exists in L. Then sup $X \cup Y$ exists in L and $\bigsqcup_L (X \cup Y) = \bigsqcup_L X \sqcup \bigsqcup_L Y$.

(4) Let L be an antisymmetric transitive relational structure with g.l.b.'s and X, Y be sets. Suppose inf X exists in L and inf Y exists in L. Then inf $X \cup Y$ exists in L and $\bigsqcap_L (X \cup Y) = \bigsqcap_L X \sqcap \bigsqcap_L Y$.

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2. Relational Substructures

The following four propositions are true:

(5) For every binary relation R and for all sets X, Y such that $X \subseteq Y$ holds $R|^2 X \subseteq R|^2 Y$.

(6) Let L be a relational structure and S, T be full relational substructures of L. Suppose the carrier of $S \subseteq$ the carrier of T. Then the internal relation of $S \subseteq$ the internal relation of T.

(7) Let L be a non empty relational structure and S be a non empty relational substructure of L. Then

(i) if X is a directed subset of S, then X is a directed subset of L, and

(ii) if X is a filtered subset of S, then X is a filtered subset of L.

(8) Let L be a non empty relational structure and S, T be non empty full relational substructures of L. Suppose the carrier of $S \subseteq$ the carrier of T. Let X be a subset of S. Then

(i) X is a subset of T, and

(ii) for every subset Y of T such that $X = Y$ holds if X is filtered, then Y is filtered and if X is directed, then Y is directed.

3. Maps

Now we present three schemes. The scheme *LambdaMD* deals with non empty relational structures $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

There exists a map f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

The scheme *KappaMD* deals with non empty relational structures $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a map f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

provided the parameters satisfy the following condition:

- For every element x of $\mathcal{A}$ holds $\mathcal{F}(x)$ is an element of $\mathcal{B}$.

The scheme *NonUniqExMD* deals with non empty relational structures $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a map f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters satisfy the following condition:

- For every element x of $\mathcal{A}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[x,y]$.

Let S, T be 1-sorted structures and let f be a map from S into T. Then $\operatorname{rng} f$ is a subset of T.

We now state the proposition

(9) Let S, T be non empty 1-sorted structures and f, g be maps from S into T. If for every element s of S holds $f(s) = g(s)$, then $f = g$.

Let J be a set, let L be a relational structure, and let f, g be functions from J into the carrier of L. The predicate $f \leq g$ is defined by:

(Def. 1) For every set j such that $j \in J$ there exist elements a, b of L such that $a = f(j)$ and $b = g(j)$ and $a \leq b$.

We introduce $g \geq f$ as a synonym of $f \leq g$.

We now state the proposition

(10) Let L, M be non empty relational structures and f, g be maps from L into M. Then $f \leq g$ if and only if for every element x of L holds $f(x) \leq g(x)$.

4. The Image of a Map

Let L, M be non empty relational structures and let f be a map from L into M. The functor $\operatorname{Im} f$ yielding a strict full relational substructure of M is defined by:

(Def. 2) $\operatorname{Im} f = \operatorname{sub}(\operatorname{rng} f)$.

We now state two propositions:

(11) For all non empty relational structures L, M and for every map f from L into M holds $\operatorname{rng} f =$ the carrier of $\operatorname{Im} f$.

(12) Let L, M be non empty relational structures, f be a map from L into M, and y be an element of $\operatorname{Im} f$. Then there exists an element x of L such that $f(x) = y$.

Let L be a non empty relational structure and let X be a non empty subset of L. Observe that $\operatorname{sub}(X)$ is non empty.

Let L, M be non empty relational structures and let f be a map from L into M. One can check that $\operatorname{Im} f$ is non empty.

5. Monotone Maps

Next we state several propositions:

(13) For every non empty relational structure L holds id_L is monotone.

(14) Let L, M, N be non empty relational structures, f be a map from L into M, and g be a map from M into N. If f is monotone and g is monotone, then $g \cdot f$ is monotone.

(15) Let L, M be non empty relational structures, f be a map from L into M, X be a subset of L, and x be an element of L. If f is monotone and $x \leq X$, then $f(x) \leq f^{\circ}X$.

(16) Let L, M be non empty relational structures, f be a map from L into M, X be a subset of L, and x be an element of L. If f is monotone and $X \leq x$, then $f^{\circ}X \leq f(x)$.

(17) Let S, T be non empty relational structures, f be a map from S into T, and X be a directed subset of S. If f is monotone, then $f^{\circ}X$ is directed.

(18) Let L be a poset with l.u.b.'s and f be a map from L into L. If f is directed-sups-preserving, then f is monotone.

(19) Let L be a poset with g.l.b.'s and f be a map from L into L. If f is filtered-infs-preserving, then f is monotone.

6. Idempotent Maps

Next we state four propositions:

(20) Let S be a non empty 1-sorted structure and f be a map from S into S. If f is idempotent, then for every element x of S holds $f(f(x)) = f(x)$.

(21) Let S be a non empty 1-sorted structure and f be a map from S into S. If f is idempotent, then $\operatorname{rng} f = \{x; x$ ranges over elements of S: $x = f(x)\}$.

(22) Let S be a non empty 1-sorted structure and f be a map from S into S. If f is idempotent, then if $X \subseteq \operatorname{rng} f$, then $f^{\circ}X = X$.

(23) For every non empty relational structure L holds id_L is idempotent.

7. Complete Lattices

In the sequel L denotes a complete lattice and a denotes an element of L.

One can prove the following propositions:

(24) If $a \in X$, then $a \leq \bigsqcup_L X$ and $\bigsqcap_L X \leq a$.

(25) Let L be a non empty relational structure. Then for every X holds sup X exists in L if and only if for every Y holds inf Y exists in L.

(26) For every non empty relational structure L such that for every X holds sup X exists in L holds L is complete.

(27) For every non empty relational structure L such that for every X holds inf X exists in L holds L is complete.

(28) Let L be a non empty relational structure. Suppose that for every subset A of L holds inf A exists in L. Let given X. Then inf X exists in L and $\bigsqcap_L X = \bigsqcap_L (X \cap \text{the carrier of } L)$.

(29) Let L be a non empty relational structure. Suppose that for every subset A of L holds sup A exists in L. Let given X. Then sup X exists in L and $\bigsqcup_L X = \bigsqcup_L (X \cap \text{the carrier of } L)$.

(30) Let L be a non empty relational structure. If for every subset A of L holds inf A exists in L, then L is complete.

Let us note that every non empty poset which is up-complete, inf-complete, and upper-bounded is also complete.

We now state several propositions:

(31) Let f be a map from L into L. Suppose f is monotone. Let M be a subset of L. If $M = \{x; x$ ranges over elements of L: $x = f(x)\}$, then $\mathrm{sub}(M)$ is a complete lattice.

(32) Every infs-inheriting non empty full relational substructure of L is a complete lattice.

(33) Every sups-inheriting non empty full relational substructure of L is a complete lattice.

(34) Let M be a non empty relational structure and f be a map from L into M. If f is sups-preserving, then Im f is sups-inheriting.

(35) Let M be a non empty relational structure and f be a map from L into M. If f is infs-preserving, then Im f is infs-inheriting.

(36) Let L, M be complete lattices and f be a map from L into M. Suppose f is sups-preserving and infs-preserving. Then Im f is a complete lattice.

(37) Let f be a map from L into L. Suppose f is idempotent and directed-sups-preserving. Then Im f is directed-sups-inheriting and Im f is a complete lattice.

8. Lattices of Ideals

We now state several propositions:

(38) Let L be a relational structure and F be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in F$ holds X is lower. Then $\bigcap F$ is a lower subset of L.

(39) Let L be a relational structure and F be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in F$ holds X is upper. Then $\bigcap F$ is an upper subset of L.

(40) Let L be an antisymmetric relational structure with l.u.b.'s and F be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in F$ holds X is lower and directed. Then $\bigcap F$ is a directed subset of L.

(41) Let L be an antisymmetric relational structure with g.l.b.'s and F be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in F$ holds X is upper and filtered. Then $\bigcap F$ is a filtered subset of L.

(42) For every poset L with g.l.b.'s and for all ideals I, J of L holds $I \cap J$ is an ideal of L.

Let L be a non empty reflexive transitive relational structure. Note that $\mathrm{Ids}(L)$ is non empty.
The following three propositions are true:

(43) Let L be a non empty reflexive transitive relational structure. Then x is an element of $\langle \mathrm{Ids}(L), \subseteq \rangle$ if and only if x is an ideal of L.

(44) Let L be a non empty reflexive transitive relational structure and I be an element of $\langle \mathrm{Ids}(L), \subseteq \rangle$. If $x \in I$, then x is an element of L.

(45) For every poset L with g.l.b.'s and for all elements x, y of $\langle \mathrm{Ids}(L), \subseteq \rangle$ holds $x \sqcap y = x \cap y$.

Let L be a poset with g.l.b.'s. Note that $\langle \mathrm{Ids}(L), \subseteq \rangle$ has g.l.b.'s.
Next we state the proposition

(46) Let L be a poset with l.u.b.'s and x, y be elements of $\langle \mathrm{Ids}(L), \subseteq \rangle$. Then there exists a subset Z of L such that

(i) $Z = \{z; z$ ranges over elements of L: $z \in x \lor z \in y \lor \bigvee_{a,b:\text{element of } L} (a \in x \land b \in y \land z = a \sqcup b)\}$,

(ii) $\sup \{x, y\}$ exists in $\langle \mathrm{Ids}(L), \subseteq \rangle$, and

(iii) $x \sqcup y = {\downarrow} Z$.

Let L be a poset with l.u.b.'s. One can check that $\langle \mathrm{Ids}(L), \subseteq \rangle$ has l.u.b.'s.
The following four propositions are true:

(47) For every lower-bounded sup-semilattice L and for every non empty subset X of $\mathrm{Ids}(L)$ holds $\bigcap X$ is an ideal of L.

(48) Let L be a lower-bounded sup-semilattice and A be a non empty subset of $\langle \mathrm{Ids}(L), \subseteq \rangle$. Then $\inf A$ exists in $\langle \mathrm{Ids}(L), \subseteq \rangle$ and $\inf A = \bigcap A$.

(49) For every poset L with l.u.b.'s holds $\inf \emptyset$ exists in $\langle \mathrm{Ids}(L), \subseteq \rangle$ and $\bigsqcap_{(\langle \mathrm{Ids}(L), \subseteq \rangle)} \emptyset = \Omega_L$.

(50) For every lower-bounded sup-semilattice L holds $\langle \mathrm{Ids}(L), \subseteq \rangle$ is complete.

Let L be a lower-bounded sup-semilattice. Note that $\langle \mathrm{Ids}(L), \subseteq \rangle$ is complete.

9. Special Maps

Let L be a non empty poset. The functor $\mathrm{SupMap}(L)$ yields a map from $\langle \mathrm{Ids}(L), \subseteq \rangle$ into L and is defined by:

(Def. 3) For every ideal I of L holds $(\mathrm{SupMap}(L))(I) = \sup I$.

Next we state three propositions:

(51) For every non empty poset L holds $\mathrm{dom}\, \mathrm{SupMap}(L) = \mathrm{Ids}(L)$ and $\mathrm{rng}\, \mathrm{SupMap}(L)$ is a subset of L.

(52) For every non empty poset L holds $x \in \mathrm{dom}\, \mathrm{SupMap}(L)$ iff x is an ideal of L.

(53) For every up-complete non empty poset L holds $\mathrm{SupMap}(L)$ is monotone.

Let L be an up-complete non empty poset. Note that $\mathrm{SupMap}(L)$ is monotone.
Let L be a non empty poset. The functor $\mathrm{IdsMap}(L)$ yields a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ and is defined by:

(Def. 4) For every element x of L holds $(\mathrm{IdsMap}(L))(x) = {\downarrow}x$.

The following proposition is true

(54) For every non empty poset L holds $\mathrm{IdsMap}(L)$ is monotone.

Let L be a non empty poset. Observe that $\mathrm{IdsMap}(L)$ is monotone.

10. THE FAMILY OF ELEMENTS IN A LATTICE

Let L be a non empty relational structure and let F be a binary relation. The functor $\bigsqcup_L F$ yields an element of L and is defined by:

(Def. 5) $\bigsqcup_L F = \bigsqcup_L \mathrm{rng} F$.

The functor $\bigsqcap_L F$ yielding an element of L is defined by:

(Def. 6) $\bigsqcap_L F = \bigsqcap_L \mathrm{rng} F$.

Let J be a set, let L be a non empty relational structure, and let F be a function from J into the carrier of L. We introduce $\mathrm{Sup}(F)$ as a synonym of $\bigsqcup_L F$. We introduce $\mathrm{Inf}(F)$ as a synonym of $\bigsqcap_L F$.

Let J be a non empty set, let S be a non empty 1-sorted structure, let F be a function from J into the carrier of S, and let j be an element of J. Then $F(j)$ is an element of S.

Let J be a set, let S be a non empty 1-sorted structure, and let F be a function from J into the carrier of S. Then $\mathrm{rng} F$ is a subset of S.

In the sequel J denotes a non empty set and j denotes an element of J.

The following propositions are true:

(55) For every function F from J into the carrier of L holds $F(j) \leq \mathrm{Sup}(F)$ and $\mathrm{Inf}(F) \leq F(j)$.

(56) For every function F from J into the carrier of L such that for every j holds $F(j) \leq a$ holds $\mathrm{Sup}(F) \leq a$.

(57) For every function F from J into the carrier of L such that for every j holds $a \leq F(j)$ holds $a \leq \mathrm{Inf}(F)$.

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