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Zermelo Theorem and Axiom of Choice

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Summary. The article is continuation of [2] and [1], and the goal of it is show that Zermelo theorem (every set has a relation which well orders it - proposition (26)) and axiom of choice (for every non-empty family of non-empty and separate sets there is set which has exactly one common element with arbitrary family member - proposition (27)) are true. It is result of the Tarski's axiom A introduced in [5] and repeated in [6]. Inclusion as a settheoretical binary relation is introduced, the correspondence of well ordering relations to ordinal numbers is shown, and basic properties of equinumerosity are presented. Some facts are based on [4].

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The articles [6], [7], [8], [3], [2], and [1] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: X, Y, Z denote sets, a, x denote sets, R denotes a binary relation, and A, B denote ordinal numbers.

Let us consider X. The functor $\subseteq_X$ yielding a binary relation is defined as follows:

(Def. 1) $\mathrm{field}(\subseteq_X) = X$ and for all Y, Z such that $Y \in X$ and $Z \in X$ holds $\langle Y, Z \rangle \in \subseteq_X$ iff $Y \subseteq Z$.

One can prove the following propositions:

(2)[1] $\subseteq_X$ is reflexive.

(3) $\subseteq_X$ is transitive.

(4) $\subseteq_A$ is connected.

(5) $\subseteq_X$ is antisymmetric.

(6) $\subseteq_A$ is well founded.

(7) $\subseteq_A$ is well-ordering.

(8) If $Y \subseteq X$, then $\subseteq_X |^2 Y = \subseteq_Y$.

(9) For all A, X such that $X \subseteq A$ holds $\subseteq_X$ is well-ordering.

(10) If $A \in B$, then $A = \subseteq_B\text{-Seg}(A)$.

(11) If $\subseteq_A$ and $\subseteq_B$ are isomorphic, then $A = B$.

(12) For all X, R, A, B such that R and $\subseteq_A$ are isomorphic and R and $\subseteq_B$ are isomorphic holds $A = B$.

[1] The proposition (1) has been removed.

(13) Let given R. Suppose R is well-ordering and for every a such that $a \in \operatorname{field} R$ there exists A such that $R|^2 R\text{-Seg}(a)$ and $\subseteq_A$ are isomorphic. Then there exists A such that R and $\subseteq_A$ are isomorphic.

(14) For every R such that R is well-ordering there exists A such that R and $\subseteq_A$ are isomorphic.

Let us consider R. Let us assume that R is well-ordering. The functor $\overline{R}$ yields an ordinal number and is defined as follows:

(Def. 2) R and $\subseteq_{\overline{R}}$ are isomorphic.

Let us consider A, R. We say that A is an order type of R if and only if:

(Def. 3) $A = \overline{R}$.

One can prove the following proposition

(17)[2] If $X \subseteq A$, then $\overline{\subseteq_X} \subseteq A$.

In the sequel f denotes a function.
Let us consider X, Y. Let us observe that $X \approx Y$ if and only if:

(Def. 4) There exists f such that f is one-to-one and $\operatorname{dom} f = X$ and $\operatorname{rng} f = Y$.

Let us notice that the predicate $X \approx Y$ is reflexive and symmetric.
We now state three propositions:

(22)[3] If $X \approx Y$ and $Y \approx Z$, then $X \approx Z$.

(25)[4] If R well orders X, then $\operatorname{field}(R|^2 X) = X$ and $R|^2 X$ is well-ordering.

(26) For every X there exists R such that R well orders X.

In the sequel M denotes a non empty set.
Next we state two propositions:

(27) Suppose for every X such that $X \in M$ holds $X \neq \emptyset$ and for all X, Y such that $X \in M$ and $Y \in M$ and $X \neq Y$ holds X misses Y. Then there exists a set C_1 such that for every X such that $X \in M$ there exists x such that $C_1 \cap X = \{x\}$.

(28) If for every X such that $X \in M$ holds $X \neq \emptyset$, then there exists a function C_1 such that $\operatorname{dom} C_1 = M$ and for every X such that $X \in M$ holds $C_1(X) \in X$.

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[2] The propositions (15) and (16) have been removed.
[3] The propositions (18)–(21) have been removed.
[4] The propositions (23) and (24) have been removed.

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