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The Theorem of Weierstrass

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Summary. The basic purpose of this article is to prove the important Weierstrass' theorem which states that a real valued continuous function f on a topological space T assumes a maximum and a minimum value on the compact subset S of T, i.e., there exist points x_1, x_2 of T being elements of S, such that $f(x_1)$ and $f(x_2)$ are the supremum and the infimum, respectively, of $f(S)$, which is the image of S under the function f. The paper is divided into three parts. In the first part, we prove some auxiliary theorems concerning properties of balls in metric spaces and define special families of subsets of topological spaces. These concepts are used in the next part of the paper which contains the essential part of the article, namely the formalization of the proof of Weierstrass' theorem. Here, we also prove a theorem concerning the compactness of images of compact sets of T under a continuous function. The final part of this work is developed for the purpose of defining some measures of the distance between compact subsets of topological metric spaces. Some simple theorems about these measures are also proved.

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The articles [18], [20], [21], [5], [6], [2], [19], [12], [1], [11], [13], [7], [14], [16], [3], [9], [15], [8], [17], [10], and [4] provide the notation and terminology for this paper.

1. Preliminaries

The following three propositions are true:

(1) Let M be a metric space, x_1, x_2 be points of M, and r_1, r_2 be real numbers. Then there exists a point x of M and there exists a real number r such that $\mathrm{Ball}(x_1,r_1)\cup\mathrm{Ball}(x_2,r_2)\subseteq\mathrm{Ball}(x,r)$.

(2) Let M be a metric space, n be a natural number, F be a family of subsets of M, and p be a finite sequence. Suppose F is finite and a family of balls and $\mathrm{rng}\,p = F$ and $\mathrm{dom}\,p = \mathrm{Seg}(n+1)$. Then there exists a family G of subsets of M such that

(i) G is finite and a family of balls, and

(ii) there exists a finite sequence q such that $\mathrm{rng}\,q = G$ and $\mathrm{dom}\,q = \mathrm{Seg}\,n$ and there exists a point x of M and there exists a real number r such that $\bigcup F \subseteq \bigcup G \cup \mathrm{Ball}(x,r)$.

(3) Let M be a metric space and F be a family of subsets of M. Suppose F is finite and a family of balls. Then there exists a point x of M and there exists a real number r such that $\bigcup F \subseteq \mathrm{Ball}(x,r)$.

Let T, S be non empty topological spaces, let f be a map from T into S, and let G be a family of subsets of S. The functor $f^{-1}(G)$ yields a family of subsets of T and is defined by:

(Def. 1) For every subset A of T holds $A \in f^{-1}(G)$ iff there exists a subset B of S such that $B \in G$ and $A = f^{-1}(B)$.

We now state two propositions:

(4) Let T, S be non empty topological spaces, f be a map from T into S, and A, B be families of subsets of S. If $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.

(5) Let T, S be non empty topological spaces, f be a map from T into S, and B be a family of subsets of S. If f is continuous and B is open, then $f^{-1}(B)$ is open.

Let T, S be non empty topological spaces, let f be a map from T into S, and let G be a family of subsets of T. The functor $f^\circ G$ yielding a family of subsets of S is defined by:

(Def. 2) For every subset A of S holds $A \in f^\circ G$ iff there exists a subset B of T such that $B \in G$ and $A = f^\circ B$.

We now state several propositions:

(6) Let T, S be non empty topological spaces, f be a map from T into S, and A, B be families of subsets of T. If $A \subseteq B$, then $f^\circ A \subseteq f^\circ B$.

(7) Let T, S be non empty topological spaces, f be a map from T into S, B be a family of subsets of S, and P be a subset of S. If $f^\circ f^{-1}(B)$ is a cover of P, then B is a cover of P.

(8) Let T, S be non empty topological spaces, f be a map from T into S, B be a family of subsets of T, and P be a subset of T. If B is a cover of P, then $f^{-1}(f^\circ B)$ is a cover of P.

(9) Let T, S be non empty topological spaces, f be a map from T into S, and Q be a family of subsets of S. Then $\bigcup(f^\circ f^{-1}(Q)) \subseteq \bigcup Q$.

(10) Let T, S be non empty topological spaces, f be a map from T into S, and P be a family of subsets of T. Then $\bigcup P \subseteq \bigcup(f^{-1}(f^\circ P))$.

(11) Let T, S be non empty topological spaces, f be a map from T into S, and Q be a family of subsets of S. If Q is finite, then $f^{-1}(Q)$ is finite.

(12) Let T, S be non empty topological spaces, f be a map from T into S, and P be a family of subsets of T. If P is finite, then $f^\circ P$ is finite.

(13) Let T, S be non empty topological spaces, f be a map from T into S, P be a subset of T, and F be a family of subsets of S. Given a family B of subsets of T such that $B \subseteq f^{-1}(F)$ and B is a cover of P and finite. Then there exists a family G of subsets of S such that $G \subseteq F$ and G is a cover of $f^\circ P$ and finite.

2. The Weierstrass' Theorem

One can prove the following propositions:

(14) Let T, S be non empty topological spaces, f be a map from T into S, and P be a subset of T. If P is compact and f is continuous, then $f^\circ P$ is compact.

(15) Let T be a non empty topological space, f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and P be a subset of T. If P is compact and f is continuous, then $f^\circ P$ is compact.

(16) Let f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. If P is compact and f is continuous, then $f^\circ P$ is compact.

Let P be a subset of $\mathbb{R}^{\mathbf{1}}$. The functor Ω_P yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 3) $\Omega_P = P$.

One can prove the following three propositions:

(17) For every subset P of $\mathbb{R}^{\mathbf{1}}$ such that P is compact holds Ω_P is bounded.

(18) For every subset P of $\mathbb{R}^{\mathbf{1}}$ such that P is compact holds Ω_P is closed.

(19) For every subset P of $\mathbb{R}^{\mathbf{1}}$ such that P is compact holds Ω_P is compact.

Let P be a subset of $\mathbb{R}^{\mathbf{1}}$. The functor $\sup P$ yielding a real number is defined as follows:

(Def. 4) $\sup P = \sup(\Omega_P)$.

The functor $\inf P$ yielding a real number is defined as follows:

(Def. 5) $\inf P = \inf(\Omega_P)$.

Next we state two propositions:

(20) Let T be a non empty topological space, f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and P be a subset of T. Suppose $P \neq \emptyset$ and P is compact and f is continuous. Then there exists a point x_1 of T such that $x_1 \in P$ and $f(x_1) = \sup(f^{\circ}P)$.

(21) Let T be a non empty topological space, f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and P be a subset of T. Suppose $P \neq \emptyset$ and P is compact and f is continuous. Then there exists a point x_2 of T such that $x_2 \in P$ and $f(x_2) = \inf(f^{\circ}P)$.

3. The Measure of the Distance Between Compact Sets

Let M be a non empty metric space and let x be a point of M. The functor $\text{dist}(x)$ yields a map from M_{top} into $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def. 6) For every point y of M holds $(\text{dist}(x))(y) = \rho(y, x)$.

Next we state three propositions:

(22) For every non empty metric space M and for every point x of M holds $\text{dist}(x)$ is continuous.

(23) Let M be a non empty metric space, x be a point of M, and P be a subset of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Then there exists a point x_1 of M_{top} such that $x_1 \in P$ and $(\text{dist}(x))(x_1) = \sup((\text{dist}(x))^{\circ}P)$.

(24) Let M be a non empty metric space, x be a point of M, and P be a subset of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Then there exists a point x_2 of M_{top} such that $x_2 \in P$ and $(\text{dist}(x))(x_2) = \inf((\text{dist}(x))^{\circ}P)$.

Let M be a non empty metric space and let P be a subset of M_{top}. The functor $\text{dist}_{\max}(P)$ yields a map from M_{top} into $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def. 7) For every point x of M holds $(\text{dist}_{\max}(P))(x) = \sup((\text{dist}(x))^{\circ}P)$.

The functor $\text{dist}_{\min}(P)$ yielding a map from M_{top} into $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 8) For every point x of M holds $(\text{dist}_{\min}(P))(x) = \inf((\text{dist}(x))^{\circ}P)$.

The following propositions are true:

(25) Let M be a non empty metric space and P be a subset of M_{top}. Suppose P is compact. Let p_1, p_2 be points of M. If $p_1 \in P$, then $\rho(p_1, p_2) \leq \sup((\text{dist}(p_2))^{\circ}P)$ and $\inf((\text{dist}(p_2))^{\circ}P) \leq \rho(p_1, p_2)$.

(26) Let M be a non empty metric space and P be a subset of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M. Then $|\sup((\text{dist}(p_1))^{\circ}P) - \sup((\text{dist}(p_2))^{\circ}P)| \leq \rho(p_1, p_2)$.

(27) Let M be a non empty metric space and P be a subset of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M and x_1, x_2 be real numbers. If $x_1 = (\mathrm{dist}_{\max}(P))(p_1)$ and $x_2 = (\mathrm{dist}_{\max}(P))(p_2)$, then $|x_1 - x_2| \leq \rho(p_1, p_2)$.

(28) Let M be a non empty metric space and P be a subset of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M. Then $|\inf((\mathrm{dist}(p_1))^{\circ}P) - \inf((\mathrm{dist}(p_2))^{\circ}P)| \leq \rho(p_1, p_2)$.

(29) Let M be a non empty metric space and P be a subset of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M and x_1, x_2 be real numbers. If $x_1 = (\mathrm{dist}_{\min}(P))(p_1)$ and $x_2 = (\mathrm{dist}_{\min}(P))(p_2)$, then $|x_1 - x_2| \leq \rho(p_1, p_2)$.

(30) For every non empty metric space M and for every subset X of $M_{\rm top}$ such that $X \neq \emptyset$ and X is compact holds $\mathrm{dist}_{\max}(X)$ is continuous.

(31) Let M be a non empty metric space and P, Q be subsets of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_1 of $M_{\rm top}$ such that $x_1 \in Q$ and $(\mathrm{dist}_{\max}(P))(x_1) = \sup((\mathrm{dist}_{\max}(P))^{\circ}Q)$.

(32) Let M be a non empty metric space and P, Q be subsets of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_2 of $M_{\rm top}$ such that $x_2 \in Q$ and $(\mathrm{dist}_{\max}(P))(x_2) = \inf((\mathrm{dist}_{\max}(P))^{\circ}Q)$.

(33) For every non empty metric space M and for every subset X of $M_{\rm top}$ such that $X \neq \emptyset$ and X is compact holds $\mathrm{dist}_{\min}(X)$ is continuous.

(34) Let M be a non empty metric space and P, Q be subsets of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_1 of $M_{\rm top}$ such that $x_1 \in Q$ and $(\mathrm{dist}_{\min}(P))(x_1) = \sup((\mathrm{dist}_{\min}(P))^{\circ}Q)$.

(35) Let M be a non empty metric space and P, Q be subsets of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_2 of $M_{\rm top}$ such that $x_2 \in Q$ and $(\mathrm{dist}_{\min}(P))(x_2) = \inf((\mathrm{dist}_{\min}(P))^{\circ}Q)$.

Let M be a non empty metric space and let P, Q be subsets of $M_{\rm top}$. The functor $\mathrm{dist}^{\min}_{\min}(P, Q)$ yields a real number and is defined as follows:

(Def. 9) $\mathrm{dist}^{\min}_{\min}(P, Q) = \inf((\mathrm{dist}_{\min}(P))^{\circ}Q)$.

The functor $\mathrm{dist}^{\max}_{\min}(P, Q)$ yielding a real number is defined by:

(Def. 10) $\mathrm{dist}^{\max}_{\min}(P, Q) = \sup((\mathrm{dist}_{\min}(P))^{\circ}Q)$.

The functor $\mathrm{dist}^{\min}_{\max}(P, Q)$ yielding a real number is defined by:

(Def. 11) $\mathrm{dist}^{\min}_{\max}(P, Q) = \inf((\mathrm{dist}_{\max}(P))^{\circ}Q)$.

The functor $\mathrm{dist}^{\max}_{\max}(P, Q)$ yielding a real number is defined as follows:

(Def. 12) $\mathrm{dist}^{\max}_{\max}(P, Q) = \sup((\mathrm{dist}_{\max}(P))^{\circ}Q)$.

We now state several propositions:

(36) Let M be a non empty metric space and P, Q be subsets of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = \mathrm{dist}^{\min}_{\min}(P, Q)$.

(37) Let M be a non empty metric space and P, Q be subsets of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = \mathrm{dist}^{\min}_{\max}(P, Q)$.

(38) Let M be a non empty metric space and P, Q be subsets of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = \mathrm{dist}^{\max}_{\min}(P, Q)$.

(39) Let M be a non empty metric space and P, Q be subsets of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = \mathrm{dist}^{\max}_{\max}(P, Q)$.

(40) Let M be a non empty metric space and P, Q be subsets of M_{top}. Suppose P is compact and Q is compact. Let x_1, x_2 be points of M. If $x_1 \in P$ and $x_2 \in Q$, then $\mathrm{dist}^{\min}_{\min}(P, Q) \leq \rho(x_1, x_2)$ and $\rho(x_1, x_2) \leq \mathrm{dist}^{\max}_{\max}(P, Q)$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[4] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/pcomps_1.html.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[7] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/compts_1.html.

[8] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/tops_2.html.

[9] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[10] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[11] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces — fundamental concepts. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topmetr.html.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/metric_1.html.

[14] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_4.html.

[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[16] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[17] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rcomp_1.html.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[19] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[20] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[21] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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