

# The “Way-Below” Relation<sup>1</sup>

Grzegorz Bancerek  
Warsaw University  
Białystok

**Summary.** In the paper the “way-below” relation, in symbols  $x \ll y$ , is introduced. Some authors prefer the term “relatively compact” or “way inside”, since in the poset of open sets of a topology it is natural to read  $U \ll V$  as “ $U$  is relatively compact in  $V$ ”. A compact element of a poset (or an element isolated from below) is defined to be way below itself. So, the compactness in the poset of open sets of a topology is equivalent to the compactness in that topology.

The article includes definitions, facts and examples 1.1–1.8 presented in [11, pp. 38–42].

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The articles [15], [19], [20], [10], [6], [7], [16], [1], [18], [17], [14], [21], [9], [8], [5], [2], [3], [12], [4], and [13] provide the notation and terminology for this paper.

## 1. THE “WAY-BELOW” RELATION

Let  $L$  be a non empty reflexive relational structure and let  $x, y$  be elements of  $L$ . We say that  $x$  is way below  $y$  if and only if:

(Def. 1) For every non empty directed subset  $D$  of  $L$  such that  $y \leq \sup D$  there exists an element  $d$  of  $L$  such that  $d \in D$  and  $x \leq d$ .

We introduce  $x \ll y$  and  $y \gg x$  as synonyms of  $x$  is way below  $y$ .

Let  $L$  be a non empty reflexive relational structure and let  $x$  be an element of  $L$ . We say that  $x$  is compact if and only if:

(Def. 2)  $x$  is way below  $x$ .

We introduce  $x$  is isolated from below as a synonym of  $x$  is compact.

Next we state several propositions:

- (1) Let  $L$  be a non empty reflexive antisymmetric relational structure and  $x, y$  be elements of  $L$ . If  $x \ll y$ , then  $x \leq y$ .
- (2) Let  $L$  be a non empty reflexive transitive relational structure and  $u, x, y, z$  be elements of  $L$ . If  $u \leq x$  and  $x \ll y$  and  $y \leq z$ , then  $u \ll z$ .
- (3) Let  $L$  be a non empty poset. Suppose  $L$  is inf-complete and has l.u.b.’s. Let  $x, y, z$  be elements of  $L$ . If  $x \ll z$  and  $y \ll z$ , then  $\sup \{x, y\}$  exists in  $L$  and  $x \sqcup y \ll z$ .

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- (4) Let  $L$  be a lower-bounded antisymmetric reflexive non empty relational structure and  $x$  be an element of  $L$ . Then  $\perp_L \ll x$ .
- (5) For every non empty poset  $L$  and for all elements  $x, y, z$  of  $L$  such that  $x \ll y$  and  $y \ll z$  holds  $x \ll z$ .
- (6) Let  $L$  be a non empty reflexive antisymmetric relational structure and  $x, y$  be elements of  $L$ . If  $x \ll y$  and  $x \gg y$ , then  $x = y$ .

Let  $L$  be a non empty reflexive relational structure and let  $x$  be an element of  $L$ . The functor  $\downarrow x$  yields a subset of  $L$  and is defined by:

(Def. 3)  $\downarrow x = \{y; y \text{ ranges over elements of } L: y \ll x\}$ .

The functor  $\uparrow x$  yielding a subset of  $L$  is defined as follows:

(Def. 4)  $\uparrow x = \{y; y \text{ ranges over elements of } L: y \gg x\}$ .

Next we state several propositions:

- (7) For every non empty reflexive relational structure  $L$  and for all elements  $x, y$  of  $L$  holds  $x \in \downarrow y$  iff  $x \ll y$ .
- (8) For every non empty reflexive relational structure  $L$  and for all elements  $x, y$  of  $L$  holds  $x \in \uparrow y$  iff  $x \gg y$ .
- (9) For every non empty reflexive antisymmetric relational structure  $L$  and for every element  $x$  of  $L$  holds  $x \geq \downarrow x$ .
- (10) For every non empty reflexive antisymmetric relational structure  $L$  and for every element  $x$  of  $L$  holds  $x \leq \uparrow x$ .
- (11) Let  $L$  be a non empty reflexive antisymmetric relational structure and  $x$  be an element of  $L$ . Then  $\downarrow x \subseteq \downarrow x$  and  $\uparrow x \subseteq \uparrow x$ .
- (12) Let  $L$  be a non empty reflexive transitive relational structure and  $x, y$  be elements of  $L$ . If  $x \leq y$ , then  $\downarrow x \subseteq \downarrow y$  and  $\uparrow y \subseteq \uparrow x$ .

Let  $L$  be a lower-bounded non empty reflexive antisymmetric relational structure and let  $x$  be an element of  $L$ . One can check that  $\downarrow x$  is non empty.

Let  $L$  be a non empty reflexive transitive relational structure and let  $x$  be an element of  $L$ . Note that  $\downarrow x$  is lower and  $\uparrow x$  is upper.

Let  $L$  be a sup-semilattice and let  $x$  be an element of  $L$ . Note that  $\downarrow x$  is directed.

Let  $L$  be an inf-complete non empty poset and let  $x$  be an element of  $L$ . Observe that  $\downarrow x$  is directed.

Let  $L$  be a connected non empty relational structure. Note that every subset of  $L$  is directed and filtered.

Let us note that every non empty chain which is up-complete and lower-bounded is also complete.

Let us note that there exists a non empty chain which is complete.

The following propositions are true:

- (13) For every up-complete non empty chain  $L$  and for all elements  $x, y$  of  $L$  such that  $x < y$  holds  $x \ll y$ .
- (14) Let  $L$  be a non empty reflexive antisymmetric relational structure and  $x, y$  be elements of  $L$ . If  $x$  is not compact and  $x \ll y$ , then  $x < y$ .
- (15) For every non empty lower-bounded reflexive antisymmetric relational structure  $L$  holds  $\perp_L$  is compact.
- (16) For every up-complete non empty poset  $L$  and for every non empty finite directed subset  $D$  of  $L$  holds  $\sup D \in D$ .
- (17) For every up-complete non empty poset  $L$  such that  $L$  is finite holds every element of  $L$  is isolated from below.

## 2. THE WAY-BELOW RELATION IN OTHER TERMS

The scheme *SSubsetEx* deals with a non empty relational structure  $\mathcal{A}$  and a unary predicate  $\mathcal{P}$ , and states that:

There exists a subset  $X$  of  $\mathcal{A}$  such that for every element  $x$  of  $\mathcal{A}$  holds  $x \in X$  iff  $\mathcal{P}[x]$  for all values of the parameters.

Next we state several propositions:

- (18) Let  $L$  be a complete lattice and  $x, y$  be elements of  $L$ . Suppose  $x \ll y$ . Let  $X$  be a subset of  $L$ . If  $y \leq \sup X$ , then there exists a finite subset  $A$  of  $L$  such that  $A \subseteq X$  and  $x \leq \sup A$ .
- (19) Let  $L$  be a complete lattice and  $x, y$  be elements of  $L$ . Suppose that for every subset  $X$  of  $L$  such that  $y \leq \sup X$  there exists a finite subset  $A$  of  $L$  such that  $A \subseteq X$  and  $x \leq \sup A$ . Then  $x \ll y$ .
- (20) Let  $L$  be a non empty reflexive transitive relational structure and  $x, y$  be elements of  $L$ . If  $x \ll y$ , then for every ideal  $I$  of  $L$  such that  $y \leq \sup I$  holds  $x \in I$ .
- (21) Let  $L$  be an up-complete non empty poset and  $x, y$  be elements of  $L$ . If for every ideal  $I$  of  $L$  such that  $y \leq \sup I$  holds  $x \in I$ , then  $x \ll y$ .
- (22) Let  $L$  be a lower-bounded lattice. Suppose  $L$  is meet-continuous. Let  $x, y$  be elements of  $L$ . Then  $x \ll y$  if and only if for every ideal  $I$  of  $L$  such that  $y = \sup I$  holds  $x \in I$ .
- (23) Let  $L$  be a complete lattice. Then every element of  $L$  is compact if and only if for every non empty subset  $X$  of  $L$  there exists an element  $x$  of  $L$  such that  $x \in X$  and for every element  $y$  of  $L$  such that  $y \in X$  holds  $x \not\ll y$ .

## 3. CONTINUOUS LATTICES

Let  $L$  be a non empty reflexive relational structure. We say that  $L$  satisfies axiom of approximation if and only if:

(Def. 5) For every element  $x$  of  $L$  holds  $x = \sup \downarrow x$ .

Let us mention that every non empty reflexive relational structure which is trivial satisfies also axiom of approximation.

Let  $L$  be a non empty reflexive relational structure. We say that  $L$  is continuous if and only if:

(Def. 6) For every element  $x$  of  $L$  holds  $\downarrow x$  is non empty and directed and  $L$  is up-complete and satisfies axiom of approximation.

Let us mention that every non empty reflexive relational structure which is continuous is also up-complete and satisfies axiom of approximation and every lower-bounded sup-semilattice which is up-complete satisfies axiom of approximation is also continuous.

Let us note that there exists a lattice which is continuous, complete, and strict.

Let  $L$  be a continuous non empty reflexive relational structure and let  $x$  be an element of  $L$ . Note that  $\downarrow x$  is non empty and directed.

We now state two propositions:

- (24) Let  $L$  be an up-complete semilattice. Suppose that for every element  $x$  of  $L$  holds  $\downarrow x$  is non empty and directed. Then  $L$  satisfies axiom of approximation if and only if for all elements  $x, y$  of  $L$  such that  $x \not\leq y$  there exists an element  $u$  of  $L$  such that  $u \ll x$  and  $u \not\leq y$ .
- (25) For every continuous lattice  $L$  and for all elements  $x, y$  of  $L$  holds  $x \leq y$  iff  $\downarrow x \subseteq \downarrow y$ .

Let us note that every non empty chain which is complete satisfies also axiom of approximation. The following proposition is true

- (26) For every complete lattice  $L$  such that every element of  $L$  is compact holds  $L$  satisfies axiom of approximation.

## 4. THE WAY-BELOW RELATION IN DIRECT POWERS

Let  $f$  be a binary relation. We say that  $f$  is nonempty if and only if:

(Def. 7) For every 1-sorted structure  $S$  such that  $S \in \text{rng } f$  holds  $S$  is non empty.

We say that  $f$  is reflexive-yielding if and only if:

(Def. 8) For every relational structure  $S$  such that  $S \in \text{rng } f$  holds  $S$  is reflexive.

Let  $I$  be a set. Note that there exists a many sorted set indexed by  $I$  which is relational structure yielding, nonempty, and reflexive-yielding.

Let  $I$  be a set and let  $J$  be a relational structure yielding nonempty many sorted set indexed by  $I$ . Observe that  $\prod J$  is non empty.

Let  $I$  be a non empty set, let  $J$  be a relational structure yielding nonempty many sorted set indexed by  $I$ , and let  $i$  be an element of  $I$ . Then  $J(i)$  is a non empty relational structure.

Let  $I$  be a set and let  $J$  be a relational structure yielding nonempty many sorted set indexed by  $I$ . Note that every element of  $\prod J$  is function-like and relation-like.

Let  $I$  be a non empty set, let  $J$  be a relational structure yielding nonempty many sorted set indexed by  $I$ , let  $x$  be an element of  $\prod J$ , and let  $i$  be an element of  $I$ . Then  $x(i)$  is an element of  $J(i)$ .

Let  $I$  be a non empty set, let  $J$  be a relational structure yielding nonempty many sorted set indexed by  $I$ , let  $i$  be an element of  $I$ , and let  $X$  be a subset of  $\prod J$ . Then  $\pi_i X$  is a subset of  $J(i)$ .

Next we state two propositions:

(27) Let  $I$  be a non empty set,  $J$  be a relational structure yielding nonempty many sorted set indexed by  $I$ , and  $x$  be a function. Then  $x$  is an element of  $\prod J$  if and only if  $\text{dom } x = I$  and for every element  $i$  of  $I$  holds  $x(i)$  is an element of  $J(i)$ .

(28) Let  $I$  be a non empty set,  $J$  be a relational structure yielding nonempty many sorted set indexed by  $I$ , and  $x, y$  be elements of  $\prod J$ . Then  $x \leq y$  if and only if for every element  $i$  of  $I$  holds  $x(i) \leq y(i)$ .

Let  $I$  be a non empty set and let  $J$  be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by  $I$ . Observe that  $\prod J$  is reflexive. Let  $i$  be an element of  $I$ . Then  $J(i)$  is a non empty reflexive relational structure.

Let  $I$  be a non empty set, let  $J$  be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by  $I$ , let  $x$  be an element of  $\prod J$ , and let  $i$  be an element of  $I$ . Then  $x(i)$  is an element of  $J(i)$ .

The following propositions are true:

(29) Let  $I$  be a non empty set and  $J$  be a relational structure yielding nonempty many sorted set indexed by  $I$ . If for every element  $i$  of  $I$  holds  $J(i)$  is transitive, then  $\prod J$  is transitive.

(30) Let  $I$  be a non empty set and  $J$  be a relational structure yielding nonempty many sorted set indexed by  $I$ . Suppose that for every element  $i$  of  $I$  holds  $J(i)$  is antisymmetric. Then  $\prod J$  is antisymmetric.

(31) Let  $I$  be a non empty set and  $J$  be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by  $I$ . Suppose that for every element  $i$  of  $I$  holds  $J(i)$  is a complete lattice. Then  $\prod J$  is a complete lattice.

(32) Let  $I$  be a non empty set and  $J$  be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by  $I$ . Suppose that for every element  $i$  of  $I$  holds  $J(i)$  is a complete lattice. Let  $X$  be a subset of  $\prod J$  and  $i$  be an element of  $I$ . Then  $(\text{sup } X)(i) = \text{sup } \pi_i X$ .

(33) Let  $I$  be a non empty set and  $J$  be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by  $I$ . Suppose that for every element  $i$  of  $I$  holds  $J(i)$  is a complete lattice. Let  $x, y$  be elements of  $\prod J$ . Then  $x \ll y$  if and only if the following conditions are satisfied:

(i) for every element  $i$  of  $I$  holds  $x(i) \ll y(i)$ , and

(ii) there exists a finite subset  $K$  of  $I$  such that for every element  $i$  of  $I$  such that  $i \notin K$  holds  $x(i) = \perp_{J(i)}$ .

## 5. THE WAY-BELOW RELATION IN TOPOLOGICAL SPACES

We now state four propositions:

- (34) Let  $T$  be a non empty topological space and  $x, y$  be elements of  $\langle \text{the topology of } T, \subseteq \rangle$ . Suppose  $x$  is way below  $y$ . Let  $F$  be a family of subsets of  $T$ . If  $F$  is open and  $y \subseteq \bigcup F$ , then there exists a finite subset  $G$  of  $F$  such that  $x \subseteq \bigcup G$ .
- (35) Let  $T$  be a non empty topological space and  $x, y$  be elements of  $\langle \text{the topology of } T, \subseteq \rangle$ . Suppose that for every family  $F$  of subsets of  $T$  such that  $F$  is open and  $y \subseteq \bigcup F$  there exists a finite subset  $G$  of  $F$  such that  $x \subseteq \bigcup G$ . Then  $x$  is way below  $y$ .
- (36) Let  $T$  be a non empty topological space,  $x$  be an element of  $\langle \text{the topology of } T, \subseteq \rangle$ , and  $X$  be a subset of  $T$ . If  $x = X$ , then  $x$  is compact iff  $X$  is compact.
- (37) Let  $T$  be a non empty topological space and  $x$  be an element of  $\langle \text{the topology of } T, \subseteq \rangle$ . Suppose  $x = \text{the carrier of } T$ . Then  $x$  is compact if and only if  $T$  is compact.

Let  $T$  be a non empty topological space. We say that  $T$  is locally-compact if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let  $x$  be a point of  $T$  and  $X$  be a subset of  $T$ . Suppose  $x \in X$  and  $X$  is open. Then there exists a subset  $Y$  of  $T$  such that  $x \in \text{Int } Y$  and  $Y \subseteq X$  and  $Y$  is compact.

Let us observe that every non empty topological space which is compact and  $T_2$  is also  $T_3$ ,  $T_4$ , and locally-compact.

One can prove the following proposition

- (38) For every set  $x$  holds  $\{\{x\}\}_{\text{top}}$  is  $T_2$ .

Let us observe that there exists a non empty topological space which is compact and  $T_2$ .

Next we state several propositions:

- (39) Let  $T$  be a non empty topological space and  $x, y$  be elements of  $\langle \text{the topology of } T, \subseteq \rangle$ . If there exists a subset  $Z$  of  $T$  such that  $x \subseteq Z$  and  $Z \subseteq y$  and  $Z$  is compact, then  $x \ll y$ .
- (40) Let  $T$  be a non empty topological space. Suppose  $T$  is locally-compact. Let  $x, y$  be elements of  $\langle \text{the topology of } T, \subseteq \rangle$ . If  $x \ll y$ , then there exists a subset  $Z$  of  $T$  such that  $x \subseteq Z$  and  $Z \subseteq y$  and  $Z$  is compact.
- (41) Let  $T$  be a non empty topological space. Suppose  $T$  is locally-compact and a  $T_2$  space. Let  $x, y$  be elements of  $\langle \text{the topology of } T, \subseteq \rangle$ . If  $x \ll y$ , then there exists a subset  $Z$  of  $T$  such that  $Z = x$  and  $\overline{Z} \subseteq y$  and  $\overline{Z}$  is compact.
- (42) Let  $X$  be a non empty topological space. Suppose  $X$  is a  $T_3$  space and  $\langle \text{the topology of } X, \subseteq \rangle$  is continuous. Then  $X$  is locally-compact.
- (43) For every non empty topological space  $T$  such that  $T$  is locally-compact holds  $\langle \text{the topology of } T, \subseteq \rangle$  is continuous.

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