

Function Spaces in the Category of Directed Suprema Preserving Maps¹

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The articles [28], [13], [34], [30], [29], [35], [10], [12], [9], [14], [1], [2], [23], [31], [4], [24], [27], [26], [11], [33], [3], [5], [6], [18], [36], [15], [20], [7], [25], [21], [32], [17], [19], [22], and [8] provide the notation and terminology for this paper.

1. CURRYING, UNCURRYING AND COMMUTING FUNCTIONS

Let F be a function. We say that F is uncurrying if and only if the conditions (Def. 1) are satisfied.

- (Def. 1)(i) For every set x such that $x \in \text{dom } F$ holds x is a function yielding function, and
(ii) for every function f such that $f \in \text{dom } F$ holds $F(f) = \text{uncurry } f$.

We say that F is currying if and only if the conditions (Def. 2) are satisfied.

- (Def. 2)(i) For every set x such that $x \in \text{dom } F$ holds x is a function and $\pi_1(x)$ is a binary relation, and
(ii) for every function f such that $f \in \text{dom } F$ holds $F(f) = \text{curry } f$.

We say that F is commuting if and only if the conditions (Def. 3) are satisfied.

- (Def. 3)(i) For every set x such that $x \in \text{dom } F$ holds x is a function yielding function, and
(ii) for every function f such that $f \in \text{dom } F$ holds $F(f) = \text{commute}(f)$.

Let us observe that every function which is empty is also uncurrying, currying, and commuting.

Let us note that there exists a function which is uncurrying, currying, and commuting.

Let F be an uncurrying function and let X be a set. One can check that $F \upharpoonright X$ is uncurrying.

Let F be a currying function and let X be a set. Note that $F \upharpoonright X$ is currying.

Next we state two propositions:

- (1) Let X, Y, Z, D be sets. Suppose $D \subseteq (Z^Y)^X$. Then there exists a many sorted set F indexed by D such that F is uncurrying and $\text{rng } F \subseteq Z^{[X, Y]}$.
- (2) Let X, Y, Z, D be sets. Suppose $D \subseteq Z^{[X, Y]}$. Then there exists a many sorted set F indexed by D such that F is currying and if $Y = \emptyset$, then $X = \emptyset$, then $\text{rng } F \subseteq (Z^Y)^X$.

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Let X, Y, Z be sets. One can verify that there exists a many sorted set indexed by $(Z^Y)^X$ which is uncurrying and there exists a many sorted set indexed by $Z^{[X, Y]}$ which is currying.

The following propositions are true:

- (3) Let A, B be non empty sets, C be a set, and f, g be commuting functions. If $\text{dom } f \subseteq (C^B)^A$ and $\text{rng } f \subseteq \text{dom } g$, then $g \cdot f = \text{id}_{\text{dom } f}$.
- (4) Let B be a non empty set, A, C be sets, f be an uncurrying function, and g be a currying function. If $\text{dom } f \subseteq (C^B)^A$ and $\text{rng } f \subseteq \text{dom } g$, then $g \cdot f = \text{id}_{\text{dom } f}$.
- (5) Let A, B, C be sets, f be a currying function, and g be an uncurrying function. If $\text{dom } f \subseteq C^{[A, B]}$ and $\text{rng } f \subseteq \text{dom } g$, then $g \cdot f = \text{id}_{\text{dom } f}$.
- (6) For every function yielding function f and for all sets i, A such that $i \in \text{dom } \text{commute}(f)$ holds $(\text{commute}(f))(i)^\circ A \subseteq \pi_i f^\circ A$.
- (7) Let f be a function yielding function and i, A be sets. If for every function g such that $g \in f^\circ A$ holds $i \in \text{dom } g$, then $\pi_i f^\circ A \subseteq (\text{commute}(f))(i)^\circ A$.
- (8) For all sets X, Y and for every function f such that $\text{rng } f \subseteq Y^X$ and for all sets i, A such that $i \in X$ holds $(\text{commute}(f))(i)^\circ A = \pi_i f^\circ A$.
- (9) For every function f and for all sets i, A such that $[A, \{i\}] \subseteq \text{dom } f$ holds $\pi_i(\text{curry } f)^\circ A = f^\circ[A, \{i\}]$.

Let X be a set and let Y be a non empty functional set. Note that every function from X into Y is function yielding.

Let T be a constituted functions 1-sorted structure. One can verify that the carrier of T is functional.

Let X be a set and let L be a non empty relational structure. Note that L^X is constituted functions.

Let us note that there exists a lattice which is constituted functions, complete, and strict and there exists a 1-sorted structure which is constituted functions and non empty.

Let T be a constituted functions non empty relational structure. Note that every non empty relational substructure of T is constituted functions.

We now state four propositions:

- (10) Let S, T be complete lattices, f be an idempotent map from T into T , and h be a map from S into $\text{Im } f$. Then $f \cdot h = h$.
- (11) Let S be a non empty relational structure and T, T_1 be non empty relational structures. Suppose T is a relational substructure of T_1 . Let f be a map from S into T and f_1 be a map from S into T_1 . If f is monotone and $f = f_1$, then f_1 is monotone.
- (12) Let S be a non empty relational structure and T, T_1 be non empty relational structures. Suppose T is a full relational substructure of T_1 . Let f be a map from S into T and f_1 be a map from S into T_1 . If f_1 is monotone and $f = f_1$, then f is monotone.
- (13) For every set X and for every subset V of X holds $(\chi_{V, X})^{-1}(\{1\}) = V$ and $(\chi_{V, X})^{-1}(\{0\}) = X \setminus V$.

2. MAPS OF POWER POSETS

Let X be a non empty set, let T be a non empty relational structure, let f be an element of T^X , and let x be an element of X . Then $f(x)$ is an element of T .

Next we state several propositions:

- (14) Let X be a non empty set, T be a non empty relational structure, and f, g be elements of T^X . Then $f \leq g$ if and only if for every element x of X holds $f(x) \leq g(x)$.

- (15) Let X be a set and L, S be non empty relational structures. Suppose the relational structure of $L =$ the relational structure of S . Then $L^X = S^X$.
- (16) Let S_1, S_2, T_1, T_2 be non empty topological spaces. Suppose that
- (i) the topological structure of $S_1 =$ the topological structure of S_2 , and
 - (ii) the topological structure of $T_1 =$ the topological structure of T_2 .
- Then $[S_1 \rightarrow T_1] = [S_2 \rightarrow T_2]$.
- (17) Let X be a set. Then there exists a map f from 2_{\subseteq}^X into $(2_{\subseteq}^1)^X$ such that f is isomorphic and for every subset Y of X holds $f(Y) = \chi_{Y,X}$.
- (18) For every set X holds 2_{\subseteq}^X and $(2_{\subseteq}^1)^X$ are isomorphic.
- (19) Let X, Y be non empty sets, T be a non empty poset, S_1 be a full non empty relational substructure of $(T^X)^Y$, S_2 be a full non empty relational substructure of $(T^Y)^X$, and F be a map from S_1 into S_2 . If F is commuting, then F is monotone.
- (20) Let X, Y be non empty sets, T be a non empty poset, S_1 be a full non empty relational substructure of $(T^Y)^X$, S_2 be a full non empty relational substructure of $T^{[X,Y]}$, and F be a map from S_1 into S_2 . If F is uncurrying, then F is monotone.
- (21) Let X, Y be non empty sets, T be a non empty poset, S_1 be a full non empty relational substructure of $(T^Y)^X$, S_2 be a full non empty relational substructure of $T^{[X,Y]}$, and F be a map from S_2 into S_1 . If F is currying, then F is monotone.

3. POSETS OF DIRECTED SUPREMA PRESERVING MAPS

Let S be a non empty relational structure and let T be a non empty reflexive antisymmetric relational structure. The functor $\text{UPS}(S, T)$ yields a strict relational structure and is defined by the conditions (Def. 4).

- (Def. 4)(i) $\text{UPS}(S, T)$ is a full relational substructure of $T^{\text{the carrier of } S}$, and
- (ii) for every set x holds $x \in$ the carrier of $\text{UPS}(S, T)$ iff x is a directed-supers-preserving map from S into T .

Let S be a non empty relational structure and let T be a non empty reflexive antisymmetric relational structure. One can check that $\text{UPS}(S, T)$ is non empty, reflexive, antisymmetric, and constituted functions.

Let S be a non empty relational structure and let T be a non empty poset. Note that $\text{UPS}(S, T)$ is transitive.

We now state the proposition

- (22) Let S be a non empty relational structure and T be a non empty reflexive antisymmetric relational structure. Then the carrier of $\text{UPS}(S, T) \subseteq$ (the carrier of T)^{the carrier of S} .

Let S be a non empty relational structure, let T be a non empty reflexive antisymmetric relational structure, let f be an element of $\text{UPS}(S, T)$, and let s be an element of S . Then $f(s)$ is an element of T .

One can prove the following propositions:

- (23) Let S be a non empty relational structure, T be a non empty reflexive antisymmetric relational structure, and f, g be elements of $\text{UPS}(S, T)$. Then $f \leq g$ if and only if for every element s of S holds $f(s) \leq g(s)$.
- (24) For all complete Scott top-lattices S, T holds $\text{UPS}(S, T) = \text{SCMaps}(S, T)$.

(25) Let S, S' be non empty relational structures and T, T' be non empty reflexive antisymmetric relational structures. Suppose that

- (i) the relational structure of $S =$ the relational structure of S' , and
- (ii) the relational structure of $T =$ the relational structure of T' .

Then $\text{UPS}(S, T) = \text{UPS}(S', T')$.

Let S, T be complete lattices. Note that $\text{UPS}(S, T)$ is complete.

One can prove the following two propositions:

(26) Let S, T be complete lattices. Then $\text{UPS}(S, T)$ is a sups-inheriting relational substructure of $T^{\text{the carrier of } S}$.

(27) For all complete lattices S, T and for every subset A of $\text{UPS}(S, T)$ holds $\text{sup}A = \bigsqcup_{(T^{\text{the carrier of } S})} A$.

Let S_1, S_2, T_1, T_2 be non empty reflexive antisymmetric relational structures and let f be a map from S_1 into S_2 . Let us assume that f is directed-sups-preserving. Let g be a map from T_1 into T_2 . Let us assume that g is directed-sups-preserving. The functor $\text{UPS}(f, g)$ yields a map from $\text{UPS}(S_2, T_1)$ into $\text{UPS}(S_1, T_2)$ and is defined by:

(Def. 5) For every directed-sups-preserving map h from S_2 into T_1 holds $(\text{UPS}(f, g))(h) = g \cdot h \cdot f$.

One can prove the following propositions:

(28) Let $S_1, S_2, S_3, T_1, T_2, T_3$ be non empty posets, f_1 be a directed-sups-preserving map from S_2 into S_3 , f_2 be a directed-sups-preserving map from S_1 into S_2 , g_1 be a directed-sups-preserving map from T_1 into T_2 , and g_2 be a directed-sups-preserving map from T_2 into T_3 . Then $\text{UPS}(f_2, g_2) \cdot \text{UPS}(f_1, g_1) = \text{UPS}(f_1 \cdot f_2, g_2 \cdot g_1)$.

(29) For all non empty reflexive antisymmetric relational structures S, T holds $\text{UPS}(\text{id}_S, \text{id}_T) = \text{id}_{\text{UPS}(S, T)}$.

(30) Let S_1, S_2, T_1, T_2 be complete lattices, f be a directed-sups-preserving map from S_1 into S_2 , and g be a directed-sups-preserving map from T_1 into T_2 . Then $\text{UPS}(f, g)$ is directed-sups-preserving.

(31) Ω (the Sierpiński space) is Scott.

(32) For every complete Scott top-lattice S holds $[S \rightarrow \text{the Sierpiński space}] = \text{UPS}(S, 2_{\subseteq}^1)$.

(33) Let S be a complete lattice. Then there exists a map F from $\text{UPS}(S, 2_{\subseteq}^1)$ into $\langle \sigma(S), \subseteq \rangle$ such that F is isomorphic and for every directed-sups-preserving map f from S into 2_{\subseteq}^1 holds $F(f) = f^{-1}(\{1\})$.

(34) For every complete lattice S holds $\text{UPS}(S, 2_{\subseteq}^1)$ and $\langle \sigma(S), \subseteq \rangle$ are isomorphic.

(35) Let S_1, S_2, T_1, T_2 be complete lattices, f be a map from S_1 into S_2 , and g be a map from T_1 into T_2 . If f is isomorphic and g is isomorphic, then $\text{UPS}(f, g)$ is isomorphic.

(36) Let S_1, S_2, T_1, T_2 be complete lattices. Suppose S_1 and S_2 are isomorphic and T_1 and T_2 are isomorphic. Then $\text{UPS}(S_2, T_1)$ and $\text{UPS}(S_1, T_2)$ are isomorphic.

(37) Let S, T be complete lattices and f be a directed-sups-preserving projection map from T into S . Then $\text{Im} \text{UPS}(\text{id}_S, f) = \text{UPS}(S, \text{Im } f)$.

(38) Let X be a non empty set, S, T be non empty posets, f be a directed-sups-preserving map from S into T^X , and i be an element of X . Then $(\text{commute}(f))(i)$ is a directed-sups-preserving map from S into T .

(39) Let X be a non empty set, S, T be non empty posets, and f be a directed-sups-preserving map from S into T^X . Then $\text{commute}(f)$ is a function from X into the carrier of $\text{UPS}(S, T)$.

- (40) Let X be a non empty set, S, T be non empty posets, and f be a function from X into the carrier of $\text{UPS}(S, T)$. Then $\text{commute}(f)$ is a directed-sups-preserving map from S into T^X .
- (41) For every non empty set X and for all non empty posets S, T holds there exists a map from $\text{UPS}(S, T^X)$ into $\text{UPS}(S, T)^X$ which is commuting and isomorphic.
- (42) For every non empty set X and for all non empty posets S, T holds $\text{UPS}(S, T^X)$ and $(\text{UPS}(S, T))^X$ are isomorphic.
- (43) For all continuous complete lattices S, T holds $\text{UPS}(S, T)$ is continuous.
- (44) For all algebraic complete lattices S, T holds $\text{UPS}(S, T)$ is algebraic.
- (45) Let R, S, T be complete lattices and f be a directed-sups-preserving map from R into $\text{UPS}(S, T)$. Then $\text{uncurry } f$ is a directed-sups-preserving map from $[:R, S:]$ into T .
- (46) Let R, S, T be complete lattices and f be a directed-sups-preserving map from $[:R, S:]$ into T . Then $\text{curry } f$ is a directed-sups-preserving map from R into $\text{UPS}(S, T)$.
- (47) For all complete lattices R, S, T holds there exists a map from $\text{UPS}(R, \text{UPS}(S, T))$ into $\text{UPS}(:R, S:], T)$ which is uncurrying and isomorphic.

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