

Lawson Topology in Continuous Lattices¹

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Summary. The article completes Mizar formalization of Section 1 of Chapter III of [11, pp. 145–147].

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The articles [16], [20], [21], [7], [8], [10], [15], [9], [19], [1], [12], [2], [22], [3], [4], [17], [14], [5], [18], [13], and [6] provide the notation and terminology for this paper.

1. SEMILATTICE HOMOMORPHISM AND INHERITANCE

Let S, T be semilattices. Let us assume that if S is upper-bounded, then T is upper-bounded. A map from S into T is said to be a semilattice morphism from S into T if:

(Def. 1) For every finite subset X of S holds it preserves inf of X .

Let S, T be semilattices. Observe that every map from S into T which is meet-preserving is also monotone.

Let S be a semilattice and let T be an upper-bounded semilattice. One can verify that every semilattice morphism from S into T is meet-preserving.

Next we state a number of propositions:

- (1) For all upper-bounded semilattices S, T and for every semilattice morphism f from S into T holds $f(\top_S) = \top_T$.
- (2) Let S, T be semilattices and f be a map from S into T . Suppose f is meet-preserving. Let X be a finite non empty subset of S . Then f preserves inf of X .
- (3) Let S, T be upper-bounded semilattices and f be a meet-preserving map from S into T . If $f(\top_S) = \top_T$, then f is a semilattice morphism from S into T .
- (4) Let S, T be semilattices and f be a map from S into T . Suppose f is meet-preserving and for every filtered non empty subset X of S holds f preserves inf of X . Let X be a non empty subset of S . Then f preserves inf of X .
- (5) Let S, T be semilattices and f be a map from S into T . Suppose f is inf-preserving. Then f is a semilattice morphism from S into T .

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12.

(6) Let S_1, T_1, S_2, T_2 be non empty relational structures. Suppose that

- (i) the relational structure of $S_1 =$ the relational structure of S_2 , and
- (ii) the relational structure of $T_1 =$ the relational structure of T_2 .

Let f_1 be a map from S_1 into T_1 and f_2 be a map from S_2 into T_2 such that $f_1 = f_2$. Then

- (iii) if f_1 is infs-preserving, then f_2 is infs-preserving, and
- (iv) if f_1 is directed-sups-preserving, then f_2 is directed-sups-preserving.

(7) Let S_1, T_1, S_2, T_2 be non empty relational structures. Suppose that

- (i) the relational structure of $S_1 =$ the relational structure of S_2 , and
- (ii) the relational structure of $T_1 =$ the relational structure of T_2 .

Let f_1 be a map from S_1 into T_1 and f_2 be a map from S_2 into T_2 such that $f_1 = f_2$. Then

- (iii) if f_1 is sups-preserving, then f_2 is sups-preserving, and
- (iv) if f_1 is filtered-infs-preserving, then f_2 is filtered-infs-preserving.

(8) Let T be a complete lattice and S be an infs-inheriting full non empty relational substructure of T . Then $\text{incl}(S, T)$ is infs-preserving.

(9) Let T be a complete lattice and S be a sups-inheriting full non empty relational substructure of T . Then $\text{incl}(S, T)$ is sups-preserving.

(10) Let T be an up-complete non empty poset and S be a directed-sups-inheriting full non empty relational substructure of T . Then $\text{incl}(S, T)$ is directed-sups-preserving.

(11) Let T be a complete lattice and S be a filtered-infs-inheriting full non empty relational substructure of T . Then $\text{incl}(S, T)$ is filtered-infs-preserving.

(12) Let T_1, T_2, R be relational structures and S be a relational substructure of T_1 . Suppose that

- (i) the relational structure of $T_1 =$ the relational structure of T_2 , and
- (ii) the relational structure of $S =$ the relational structure of R .

Then R is a relational substructure of T_2 and if S is full, then R is a full relational substructure of T_2 .

(13) Every non empty relational structure T is an infs-inheriting sups-inheriting full relational substructure of T .

Let T be a complete lattice. Observe that there exists a continuous subframe of T which is complete.

We now state a number of propositions:

(14) Let T be a semilattice and S be a full non empty relational substructure of T . Then S is meet-inheriting if and only if for every finite non empty subset X of S holds $\bigcap_T X \in$ the carrier of S .

(15) Let T be a sup-semilattice and S be a full non empty relational substructure of T . Then S is join-inheriting if and only if for every finite non empty subset X of S holds $\bigcup_T X \in$ the carrier of S .

(16) Let T be an upper-bounded semilattice and S be a meet-inheriting full non empty relational substructure of T . Suppose $\top_T \in$ the carrier of S and S is filtered-infs-inheriting. Then S is infs-inheriting.

(17) Let T be a lower-bounded sup-semilattice and S be a join-inheriting full non empty relational substructure of T . Suppose $\perp_T \in$ the carrier of S and S is directed-sups-inheriting. Then S is sups-inheriting.

- (18) Let T be a complete lattice and S be a full non empty relational substructure of T . If S is infs-inheriting, then S is complete.
- (19) Let T be a complete lattice and S be a full non empty relational substructure of T . If S is sups-inheriting, then S is complete.
- (20) Let T_1, T_2 be non empty relational structures, S_1 be a non empty full relational substructure of T_1 , and S_2 be a non empty full relational substructure of T_2 . Suppose that
- (i) the relational structure of $T_1 =$ the relational structure of T_2 , and
 - (ii) the carrier of $S_1 =$ the carrier of S_2 .
- If S_1 is infs-inheriting, then S_2 is infs-inheriting.
- (21) Let T_1, T_2 be non empty relational structures, S_1 be a non empty full relational substructure of T_1 , and S_2 be a non empty full relational substructure of T_2 . Suppose that
- (i) the relational structure of $T_1 =$ the relational structure of T_2 , and
 - (ii) the carrier of $S_1 =$ the carrier of S_2 .
- If S_1 is sups-inheriting, then S_2 is sups-inheriting.
- (22) Let T_1, T_2 be non empty relational structures, S_1 be a non empty full relational substructure of T_1 , and S_2 be a non empty full relational substructure of T_2 . Suppose that
- (i) the relational structure of $T_1 =$ the relational structure of T_2 , and
 - (ii) the carrier of $S_1 =$ the carrier of S_2 .
- If S_1 is directed-sups-inheriting, then S_2 is directed-sups-inheriting.
- (23) Let T_1, T_2 be non empty relational structures, S_1 be a non empty full relational substructure of T_1 , and S_2 be a non empty full relational substructure of T_2 . Suppose that
- (i) the relational structure of $T_1 =$ the relational structure of T_2 , and
 - (ii) the carrier of $S_1 =$ the carrier of S_2 .
- If S_1 is filtered-infs-inheriting, then S_2 is filtered-infs-inheriting.

2. NETS AND LIMITS

We now state the proposition

- (24) Let S, T be non empty topological spaces, N be a net in S , and f be a map from S into T . If f is continuous, then $f^\circ \text{Lim } N \subseteq \text{Lim}(f \cdot N)$.

Let T be a non empty relational structure and let N be a non empty net structure over T . Let us observe that N is antitone if and only if:

- (Def. 2) For all elements i, j of N such that $i \leq j$ holds $N(i) \geq N(j)$.

Let T be a non empty reflexive relational structure and let x be an element of T . Note that $\langle \{x\}^{\text{op}}; \text{id} \rangle$ is transitive, directed, monotone, and antitone.

Let T be a non empty reflexive relational structure. Observe that there exists a net in T which is monotone, antitone, reflexive, and strict.

Let T be a non empty relational structure and let F be a non empty subset of T . Observe that $\langle F^{\text{op}}; \text{id} \rangle$ is antitone.

Let S, T be non empty reflexive relational structures, let f be a monotone map from S into T , and let N be an antitone non empty net structure over S . Note that $f \cdot N$ is antitone.

One can prove the following propositions:

- (25) Let S be a complete lattice and N be a net in S . Then $\{\bigcap_S \{N(i); i \text{ ranges over elements of } N: i \geq j\} : j \text{ ranges over elements of } N\}$ is a directed non empty subset of S .

- (26) Let S be a non empty poset and N be a monotone reflexive net in S . Then $\{\bigcap_S \{N(i); i \text{ ranges over elements of } N: i \geq j\} : j \text{ ranges over elements of } N\}$ is a directed non empty subset of S .
- (27) Let S be a non empty 1-sorted structure, N be a non empty net structure over S , and X be a set. If $\text{rng}(\text{the mapping of } N) \subseteq X$, then N is eventually in X .
- (28) For every inf-complete non empty poset R and for every non empty filtered subset F of R holds $\liminf \langle F^{\text{op}}; \text{id} \rangle = \inf F$.
- (29) Let S, T be inf-complete non empty posets, X be a non empty filtered subset of S , and f be a monotone map from S into T . Then $\liminf \langle f \cdot \langle X^{\text{op}}; \text{id} \rangle \rangle = \inf \langle f \circ X \rangle$.
- (30) Let S, T be non empty top-posets, X be a non empty filtered subset of S , f be a monotone map from S into T , and Y be a non empty filtered subset of T . If $Y = f \circ X$, then $f \cdot \langle X^{\text{op}}; \text{id} \rangle$ is a subnet of $\langle Y^{\text{op}}; \text{id} \rangle$.
- (31) Let S, T be non empty top-posets, X be a non empty filtered subset of S , f be a monotone map from S into T , and Y be a non empty filtered subset of T . If $Y = f \circ X$, then $\text{Lim} \langle Y^{\text{op}}; \text{id} \rangle \subseteq \text{Lim} \langle f \cdot \langle X^{\text{op}}; \text{id} \rangle \rangle$.
- (32) Let S be a non empty reflexive relational structure and D be a non empty subset of S . Then the mapping of $\text{NetStr}(D) = \text{id}_D$ and the carrier of $\text{NetStr}(D) = D$ and $\text{NetStr}(D)$ is a full relational substructure of S .
- (33) Let S, T be up-complete non empty posets, f be a monotone map from S into T , and D be a non empty directed subset of S . Then $\liminf \langle f \cdot \text{NetStr}(D) \rangle = \sup \langle f \circ D \rangle$.
- (34) Let S be a non empty reflexive relational structure, D be a non empty directed subset of S , and i, j be elements of $\text{NetStr}(D)$. Then $i \leq j$ if and only if $(\text{NetStr}(D))(i) \leq (\text{NetStr}(D))(j)$.
- (35) For every Lawson complete top-lattice T and for every directed non empty subset D of T holds $\sup D \in \text{LimNetStr}(D)$.

Let T be a non empty 1-sorted structure, let N be a net in T , and let M be a non empty net structure over T . Let us assume that M is a subnet of N . A map from M into N is said to be an embedding of M into N if it satisfies the conditions (Def. 3).

- (Def. 3)(i) The mapping of $M = (\text{the mapping of } N) \cdot \text{it}$, and
- (ii) for every element m of N there exists an element n of M such that for every element p of M such that $n \leq p$ holds $m \leq \text{it}(p)$.

One can prove the following propositions:

- (36) Let T be a non empty 1-sorted structure, N be a net in T , M be a non empty subnet of N , e be an embedding of M into N , and i be an element of M . Then $M(i) = N(e(i))$.
- (37) For every complete lattice T and for every net N in T and for every subnet M of N holds $\liminf N \leq \liminf M$.
- (38) Let T be a complete lattice, N be a net in T , M be a subnet of N , and e be an embedding of M into N . Suppose that for every element i of N and for every element j of M such that $e(j) \leq i$ there exists an element j' of M such that $j' \geq j$ and $N(i) \geq M(j')$. Then $\liminf N = \liminf M$.
- (39) Let T be a non empty relational structure, N be a net in T , and M be a non empty full structure of a subnet of N . Suppose that for every element i of N there exists an element j of N such that $j \geq i$ and $j \in \text{the carrier of } M$. Then M is a subnet of N and $\text{incl}(M, N)$ is an embedding of M into N .
- (40) Let T be a non empty relational structure, N be a net in T , and i be an element of N . Then $N|i$ is a subnet of N and $\text{incl}(N|i, N)$ is an embedding of $N|i$ into N .

- (41) For every complete lattice T and for every net N in T and for every element i of N holds $\liminf(N \upharpoonright i) = \liminf N$.
- (42) Let T be a non empty relational structure, N be a net in T , and X be a set. Suppose N is eventually in X . Then there exists an element i of N such that $N(i) \in X$ and $\text{rng}(\text{the mapping of } N \upharpoonright i) \subseteq X$.
- (43) Let T be a Lawson complete top-lattice and N be an eventually-filtered net in T . Then $\text{rng}(\text{the mapping of } N)$ is a filtered non empty subset of T .
- (44) For every Lawson complete top-lattice T and for every eventually-filtered net N in T holds $\text{Lim}N = \{\inf N\}$.

3. LAWSON TOPOLOGY REVISITED

The following propositions are true:

- (45) Let S, T be Lawson complete top-lattices and f be a meet-preserving map from S into T . Then f is continuous if and only if the following conditions are satisfied:
- (i) f is directed-sups-preserving, and
 - (ii) for every non empty subset X of S holds f preserves inf of X .
- (46) Let S, T be Lawson complete top-lattices and f be a semilattice morphism from S into T . Then f is continuous if and only if f is infs-preserving and directed-sups-preserving.

Let S, T be non empty relational structures and let f be a map from S into T . We say that f is liminfs -preserving if and only if:

(Def. 4) For every net N in S holds $f(\liminf N) = \liminf(f \cdot N)$.

The following propositions are true:

- (47) Let S, T be Lawson complete top-lattices and f be a semilattice morphism from S into T . Then f is continuous if and only if f is liminfs -preserving.
- (48) Let T be a Lawson complete continuous top-lattice and S be a meet-inheriting full non empty relational substructure of T . Suppose $\top_T \in$ the carrier of S and there exists a subset X of T such that $X =$ the carrier of S and X is closed. Then S is infs-inheriting.
- (49) Let T be a Lawson complete continuous top-lattice and S be a full non empty relational substructure of T . Given a subset X of T such that $X =$ the carrier of S and X is closed. Then S is directed-sups-inheriting.
- (50) Let T be a Lawson complete continuous top-lattice and S be an infs-inheriting directed-sups-inheriting full non empty relational substructure of T . Then there exists a subset X of T such that $X =$ the carrier of S and X is closed.
- (51) Let T be a Lawson complete continuous top-lattice, S be an infs-inheriting directed-sups-inheriting full non empty relational substructure of T , and N be a net in T . If N is eventually in the carrier of S , then $\liminf N \in$ the carrier of S .
- (52) Let T be a Lawson complete continuous top-lattice and S be a meet-inheriting full non empty relational substructure of T . Suppose that
- (i) $\top_T \in$ the carrier of S , and
 - (ii) for every net N in T such that $\text{rng}(\text{the mapping of } N) \subseteq$ the carrier of S holds $\liminf N \in$ the carrier of S .

Then S is infs-inheriting.

- (53) Let T be a Lawson complete continuous top-lattice and S be a full non empty relational substructure of T . Suppose that for every net N in T such that $\text{rng}(N) \subseteq$ the carrier of S holds $\liminf N \in$ the carrier of S . Then S is directed-sups-inheriting.
- (54) Let T be a Lawson complete continuous top-lattice, S be a meet-inheriting full non empty relational substructure of T , and X be a subset of T . Suppose $X =$ the carrier of S and $\top_T \in X$. Then X is closed if and only if for every net N in T such that N is eventually in X holds $\liminf N \in X$.

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