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Algebraic and Arithmetic Lattices. Part II[1]

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Summary. The article is a translation of [10, pp. 89–92].

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The articles [16], [18], [19], [6], [7], [9], [17], [15], [2], [14], [1], [3], [11], [21], [4], [8], [5], [20], [12], and [13] provide the notation and terminology for this paper.

1. PRELIMINARIES

One can prove the following propositions:

(1) Let R be a relational structure and S be a full relational substructure of R. Then every full relational substructure of S is a full relational substructure of R.

(2) Let X be a 1-sorted structure, Y, Z be non empty 1-sorted structures, f be a map from X into Y, and g be a map from Y into Z. If f is onto and g is onto, then $g \cdot f$ is onto.

(3) For every 1-sorted structure X and for every subset Y of X holds $(\mathrm{id}_X)^\circ Y = Y$.

(4) For every set X and for every element a of $2^X_\subseteq$ holds $\uparrow a = \{Y; Y$ ranges over subsets of X: $a \subseteq Y\}$.

(5) Let L be an upper-bounded non empty antisymmetric relational structure and a be an element of L. If $\top_L \leq a$, then $a = \top_L$.

(6) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. If g is onto and $\langle g, d\rangle$ is Galois, then T and $\mathrm{Im}\, d$ are isomorphic.

(7) Let L_1, L_2, L_3 be non empty posets, g_1 be a map from L_1 into L_2, g_2 be a map from L_2 into L_3, d_1 be a map from L_2 into L_1, and d_2 be a map from L_3 into L_2. If $\langle g_1, d_1\rangle$ is Galois and $\langle g_2, d_2\rangle$ is Galois, then $\langle g_2 \cdot g_1, d_1 \cdot d_2\rangle$ is Galois.

(8) Let L_1, L_2 be non empty posets, f be a map from L_1 into L_2, and f_1 be a map from L_2 into L_1. Suppose $f_1 = (f$ **qua** function$)^{-1}$ and f is isomorphic. Then $\langle f, f_1\rangle$ is Galois and $\langle f_1, f\rangle$ is Galois.

(9) For every set X holds $2^X_\subseteq$ is arithmetic.

Let X be a set. Observe that $2^X_\subseteq$ is arithmetic.

Next we state four propositions:

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(10) Let L_1, L_2 be up-complete non empty posets and f be a map from L_1 into L_2. If f is isomorphic, then for every element x of L_1 holds $f^\circ{\downarrow}x = {\downarrow}f(x)$.

(11) For all non empty posets L_1, L_2 such that L_1 and L_2 are isomorphic and L_1 is continuous holds L_2 is continuous.

(12) Let L_1, L_2 be lattices. Suppose L_1 and L_2 are isomorphic and L_1 is lower-bounded and arithmetic. Then L_2 is arithmetic.

(13) Let L_1, L_2, L_3 be non empty posets, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is directed-sups-preserving and g is directed-sups-preserving. Then $g \cdot f$ is directed-sups-preserving.

2. Maps Preserving Sup's and Inf's

Next we state several propositions:

(14) Let L_1, L_2 be non empty relational structures, f be a map from L_1 into L_2, and X be a subset of $\mathrm{Im}\, f$. Then $(f_\circ)^\circ X = X$.

(15) Let X be a set and c be a map from $2^X_\subseteq$ into $2^X_\subseteq$. Suppose c is idempotent and directed-sups-preserving. Then $c_\circ$ is directed-sups-preserving.

(16) Let L be a continuous complete lattice and p be a kernel map from L into L. If p is directed-sups-preserving, then $\mathrm{Im}\, p$ is a continuous lattice.

(17) Let L be a continuous complete lattice and p be a projection map from L into L. If p is directed-sups-preserving, then $\mathrm{Im}\, p$ is a continuous lattice.

(18) Let L be a lower-bounded lattice. Then L is continuous if and only if there exists an arithmetic lower-bounded lattice A such that there exists a map from A into L which is onto, infs-preserving, and directed-sups-preserving.

(19) Let L be a lower-bounded lattice. Then L is continuous if and only if there exists an algebraic lower-bounded lattice A such that there exists a map from A into L which is onto, infs-preserving, and directed-sups-preserving.

(20) Let L be a lower-bounded lattice. Then

(i) if L is continuous, then there exists a non empty set X and there exists a projection map p from $2^X_\subseteq$ into $2^X_\subseteq$ such that p is directed-sups-preserving and L and $\mathrm{Im}\, p$ are isomorphic, and

(ii) if there exists a set X and there exists a projection map p from $2^X_\subseteq$ into $2^X_\subseteq$ such that p is directed-sups-preserving and L and $\mathrm{Im}\, p$ are isomorphic, then L is continuous.

3. Atoms Elements

The following proposition is true

(21) For every non empty relational structure L and for every element x of L holds $x \in \mathrm{PRIME}(L^{\mathrm{op}})$ iff x is co-prime.

Let L be a non empty relational structure and let a be an element of L. We say that a is atom if and only if:

(Def. 1) $\bot_L < a$ and for every element b of L such that $\bot_L < b$ and $b \leq a$ holds $b = a$.

Let L be a non empty relational structure. The functor $\mathrm{ATOM}(L)$ yielding a subset of L is defined as follows:

(Def. 2) For every element x of L holds $x \in \mathrm{ATOM}(L)$ iff x is atom.

We now state the proposition

(23)[1] For every Boolean lattice L and for every element a of L holds a is atom iff a is co-prime and $a \neq \bot_L$.

Let L be a Boolean lattice. Observe that every element of L which is atom is also co-prime.
The following propositions are true:

(24) For every Boolean lattice L holds $\mathrm{ATOM}(L) = \mathrm{PRIME}(L^{\mathrm{op}}) \setminus \{\bot_L\}$.

(25) For every Boolean lattice L and for all elements x, a of L such that a is atom holds $a \leq x$ iff $a \not\leq \neg x$.

(26) Let L be a complete Boolean lattice, X be a subset of L, and x be an element of L. Then $x \sqcap \sup X = \bigsqcup_L \{x \sqcap y; y$ ranges over elements of L: $y \in X\}$.

(27) Let L be a lower-bounded antisymmetric non empty relational structure with g.l.b.'s and x, y be elements of L. If x is atom and y is atom and $x \neq y$, then $x \sqcap y = \bot_L$.

(28) Let L be a complete Boolean lattice, x be an element of L, and A be a subset of L. If $A \subseteq \mathrm{ATOM}(L)$, then $x \in A$ iff x is atom and $x \leq \sup A$.

(29) Let L be a complete Boolean lattice and X, Y be subsets of L. If $X \subseteq \mathrm{ATOM}(L)$ and $Y \subseteq \mathrm{ATOM}(L)$, then $X \subseteq Y$ iff $\sup X \leq \sup Y$.

4. More on the Boolean Lattice

The following propositions are true:

(30) For every Boolean lattice L holds L is arithmetic iff there exists a set X such that L and $2^X_\subseteq$ are isomorphic.

(31) For every Boolean lattice L holds L is arithmetic iff L is algebraic.

(32) For every Boolean lattice L holds L is arithmetic iff L is continuous.

(33) For every Boolean lattice L holds L is arithmetic iff L is continuous and L^{op} is continuous.

(34) For every Boolean lattice L holds L is arithmetic iff L is completely-distributive.

(35) Let L be a Boolean lattice. Then L is arithmetic if and only if the following conditions are satisfied:

(i) L is complete, and

(ii) for every element x of L there exists a subset X of L such that $X \subseteq \mathrm{ATOM}(L)$ and $x = \sup X$.

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