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The Steinitz Theorem and the Dimension of a Vector Space

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Summary. The main purpose of the paper is to define the dimension of an abstract vector space. The dimension of a finite-dimensional vector space is, by the most common definition, the number of vectors in a basis. Obviously, each basis contains the same number of vectors. We prove the Steinitz Theorem together with Exchange Lemma in the second section. The Steinitz Theorem says that each linearly-independent subset of a vector space has cardinality less than any subset that generates the space, moreover it can be extended to a basis. Further we review some of the standard facts involving the dimension of a vector space. Additionally, in the last section, we introduce two notions: the family of subspaces of a fixed dimension and the pencil of subspaces. Both of them can be applied in the algebraic representation of several geometries.

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The articles [10], [18], [11], [2], [19], [4], [5], [1], [6], [3], [16], [7], [12], [8], [17], [14], [15], [13], and [9] provide the notation and terminology for this paper.

1. PRELIMINARIES

For simplicity, we adopt the following rules: G_1 is a field, V is a vector space over G_1, W is a subspace of V, x is a set, and n is a natural number.

Let S be a non empty 1-sorted structure. Note that there exists a subset of S which is non empty.

One can prove the following proposition

(1) For every finite set X such that $n \leq \overline{\overline{X}}$ there exists a finite subset A of X such that $\overline{\overline{A}} = n$.

In the sequel f, g are functions.

One can prove the following propositions:

(2) For every f such that f is one-to-one holds if $x \in \operatorname{rng} f$, then $\overline{\overline{f^{-1}(\{x\})}} = 1$.

(3) For every f such that $x \notin \operatorname{rng} f$ holds $\overline{\overline{f^{-1}(\{x\})}} = 0$.

(4) For all f, g such that $\operatorname{rng} f = \operatorname{rng} g$ and f is one-to-one and g is one-to-one holds f and g are fiberwise equipotent.

(5) Let L be a linear combination of V, F, G be finite sequences of elements of the carrier of V, and P be a permutation of $\operatorname{dom} F$. If $G = F \cdot P$, then $\sum(L\, F) = \sum(L\, G)$.

(6) Let L be a linear combination of V and F be a finite sequence of elements of the carrier of V. If the support of L misses $\operatorname{rng} F$, then $\sum(L\,F) = 0_V$.

(7) Let F be a finite sequence of elements of the carrier of V. Suppose F is one-to-one. Let L be a linear combination of V. If the support of $L \subseteq \operatorname{rng} F$, then $\sum(L\,F) = \sum L$.

(8) Let L be a linear combination of V and F be a finite sequence of elements of the carrier of V. Then there exists a linear combination K of V such that the support of $K = \operatorname{rng} F \cap$ (the support of L) and $L\,F = K\,F$.

(9) Let L be a linear combination of V, A be a subset of V, and F be a finite sequence of elements of the carrier of V. Suppose $\operatorname{rng} F \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum(L\,F) = \sum K$.

(10) Let L be a linear combination of V and A be a subset of V. Suppose the support of $L \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum L = \sum K$.

(11) Let L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Let K be a linear combination of W. Suppose $K = L \restriction$the carrier of W. Then the support of $L =$ the support of K and $\sum L = \sum K$.

(12) Let K be a linear combination of W. Then there exists a linear combination L of V such that the support of $K =$ the support of L and $\sum K = \sum L$.

(13) Let L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Then there exists a linear combination K of W such that the support of $K =$ the support of L and $\sum K = \sum L$.

(14) For every basis I of V and for every vector v of V holds $v \in \operatorname{Lin}(I)$.

(15) Let A be a subset of W. Suppose A is linearly independent. Then there exists a subset B of V such that B is linearly independent and $B = A$.

(16) Let A be a subset of V. Suppose A is linearly independent and $A \subseteq$ the carrier of W. Then there exists a subset B of W such that B is linearly independent and $B = A$.

(17) For every basis A of W there exists a basis B of V such that $A \subseteq B$.

(18) Let A be a subset of V. Suppose A is linearly independent. Let v be a vector of V. If $v \in A$, then for every subset B of V such that $B = A \setminus \{v\}$ holds $v \notin \operatorname{Lin}(B)$.

(19) Let I be a basis of V and A be a non empty subset of V. Suppose A misses I. Let B be a subset of V. If $B = I \cup A$, then B is linearly dependent.

(20) For every subset A of V such that $A \subseteq$ the carrier of W holds $\operatorname{Lin}(A)$ is a subspace of W.

(21) For every subset A of V and for every subset B of W such that $A = B$ holds $\operatorname{Lin}(A) = \operatorname{Lin}(B)$.

2. The Steinitz Theorem

One can prove the following propositions:

(22) Let A, B be finite subsets of V and v be a vector of V. Suppose $v \in \operatorname{Lin}(A \cup B)$ and $v \notin \operatorname{Lin}(B)$. Then there exists a vector w of V such that $w \in A$ and $w \in \operatorname{Lin}(((A \cup B) \setminus \{w\}) \cup \{v\})$.

(23) Let A, B be finite subsets of V. Suppose the vector space structure of $V = \operatorname{Lin}(A)$ and B is linearly independent. Then $\overline{\overline{B}} \leq \overline{\overline{A}}$ and there exists a finite subset C of V such that $C \subseteq A$ and $\overline{\overline{C}} = \overline{\overline{A}} - \overline{\overline{B}}$ and the vector space structure of $V = \operatorname{Lin}(B \cup C)$.

3. Finite-Dimensional Vector Spaces

Let G_1 be a field and let V be a vector space over G_1. Let us observe that V is finite dimensional if and only if:

(Def. 1) There exists a finite subset of V which is a basis of V.

We now state several propositions:

(24) If V is finite dimensional, then every basis of V is finite.

(25) If V is finite dimensional, then for every subset A of V such that A is linearly independent holds A is finite.

(26) If V is finite dimensional, then for all bases A, B of V holds $\overline{\overline{A}} = \overline{\overline{B}}$.

(27) $\mathbf{0}_V$ is finite dimensional.

(28) If V is finite dimensional, then W is finite dimensional.

Let G_1 be a field and let V be a vector space over G_1. One can check that there exists a subspace of V which is strict and finite dimensional.

Let G_1 be a field and let V be a finite dimensional vector space over G_1. One can check that every subspace of V is finite dimensional.

Let G_1 be a field and let V be a finite dimensional vector space over G_1. Note that there exists a subspace of V which is strict.

4. The Dimension of a Vector Space

Let G_1 be a field and let V be a vector space over G_1. Let us assume that V is finite dimensional. The functor $\dim(V)$ yields a natural number and is defined by:

(Def. 2) For every basis I of V holds $\dim(V) = \overline{\overline{I}}$.

We adopt the following rules: V denotes a finite dimensional vector space over G_1, W, W_1, W_2 denote subspaces of V, and u, v denote vectors of V.

Next we state a number of propositions:

(29) $\dim(W) \leq \dim(V)$.

(30) For every subset A of V such that A is linearly independent holds $\overline{\overline{A}} = \dim(\mathrm{Lin}(A))$.

(31) $\dim(V) = \dim(\Omega_V)$.

(32) $\dim(V) = \dim(W)$ iff $\Omega_V = \Omega_W$.

(33) $\dim(V) = 0$ iff $\Omega_V = \mathbf{0}_V$.

(34) $\dim(V) = 1$ iff there exists v such that $v \neq 0_V$ and $\Omega_V = \mathrm{Lin}(\{v\})$.

(35) $\dim(V) = 2$ iff there exist u, v such that $u \neq v$ and $\{u, v\}$ is linearly independent and $\Omega_V = \mathrm{Lin}(\{u, v\})$.

(36) $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

(37) $\dim(W_1 \cap W_2) \geq (\dim(W_1) + \dim(W_2)) - \dim(V)$.

(38) If V is the direct sum of W_1 and W_2, then $\dim(V) = \dim(W_1) + \dim(W_2)$.

(39) $n \leq \dim(V)$ iff there exists a strict subspace W of V such that $\dim(W) = n$.

Let G_1 be a field, let V be a finite dimensional vector space over G_1, and let n be a natural number. The functor $\mathrm{Sub}_n(V)$ yields a set and is defined by:

(Def. 3) $x \in \mathrm{Sub}_n(V)$ iff there exists a strict subspace W of V such that $W = x$ and $\dim(W) = n$.

Next we state three propositions:

(40) If $n \leq \dim(V)$, then $\mathrm{Sub}_n(V)$ is non empty.

(41) If $\dim(V) < n$, then $\mathrm{Sub}_n(V) = \emptyset$.

(42) $\mathrm{Sub}_n(W) \subseteq \mathrm{Sub}_n(V)$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[6] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[7] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfinseq.html.

[8] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[9] Robert Milewski. Associated matrix of linear map. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/matrlin.html.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[11] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[12] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[13] Wojciech A. Trybulec. Basis of vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_7.html.

[14] Wojciech A. Trybulec. Linear combinations in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_6.html.

[15] Wojciech A. Trybulec. Operations on subspaces in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_5.html.

[16] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[17] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_4.html.

[18] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[19] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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