On the Lattice of Subspaces of a Vector Space

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The articles [10], [5], [15], [6], [9], [11], [7], [17], [13], [1], [14], [16], [2], [4], [3], [12], and [8] provide the notation and terminology for this paper.

In this paper F is a field and V_1 is a strict vector space over F.

Let us consider F, V_1. The functor $\mathbb{L}_{(V_1)}$ yields a strict bounded lattice and is defined as follows:

(Def. 1) $\mathbb{L}_{(V_1)} = \langle \mathrm{Subspaces}\, V_1, \mathrm{SubJoin}\, V_1, \mathrm{SubMeet}\, V_1 \rangle$.

Let us consider F, V_1. Family of subspaces of V_1 is defined by:

(Def. 2)   For every set x such that $x \in$ it holds x is a subspace of V_1.

Let us consider F, V_1. Note that there exists a family of subspaces of V_1 which is non empty.

Let us consider F, V_1. Then $\mathrm{Subspaces}\, V_1$ is a non empty family of subspaces of V_1. Let X be a non empty family of subspaces of V_1. We see that the element of X is a subspace of V_1.

Let us consider F, V_1 and let x be an element of $\mathrm{Subspaces}\, V_1$. The functor $\bar{x}$ yields a subset of V_1 and is defined as follows:

(Def. 3)   There exists a subspace X of V_1 such that $x = X$ and $\bar{x} =$ the carrier of X.

Let us consider F, V_1. The functor $\overline{V_1}$ yielding a function from $\mathrm{Subspaces}\, V_1$ into $2^{\text{the carrier of } V_1}$ is defined by:

(Def. 4)   For every element h of $\mathrm{Subspaces}\, V_1$ and for every subspace H of V_1 such that $h = H$ holds $\overline{V_1}(h) =$ the carrier of H.

The following propositions are true:

(1)   For every strict vector space V_1 over F and for every non empty subset H of $\mathrm{Subspaces}\, V_1$ holds $\overline{V_1}^{\circ} H$ is non empty.

(2)   For every strict vector space V_1 over F and for every strict subspace H of V_1 holds $0_{(V_1)} \in \overline{V_1}(H)$.

Let us consider F, V_1 and let G be a non empty subset of $\mathrm{Subspaces}\, V_1$. The functor $\bigcap G$ yielding a strict subspace of V_1 is defined as follows:

(Def. 5)   The carrier of $\bigcap G = \bigcap (\overline{V_1}^{\circ} G)$.

We now state several propositions:

(3)   Subspaces V_1 = the carrier of $\mathbb{L}_{(V_1)}$.

(4)   The meet operation of $\mathbb{L}_{(V_1)}$ = SubMeet V_1.

(5)   The join operation of $\mathbb{L}_{(V_1)}$ = SubJoin V_1.

(6)   Let V_1 be a strict vector space over F, p, q be elements of $\mathbb{L}_{(V_1)}$, and H_1, H_2 be strict subspaces of V_1. Suppose $p = H_1$ and $q = H_2$. Then $p \sqsubseteq q$ if and only if the carrier of $H_1 \subseteq$ the carrier of H_2.

(7)   Let V_1 be a strict vector space over F, p, q be elements of $\mathbb{L}_{(V_1)}$, and H_1, H_2 be subspaces of V_1. If $p = H_1$ and $q = H_2$, then $p \sqcup q = H_1 + H_2$.

(8)   Let V_1 be a strict vector space over F, p, q be elements of $\mathbb{L}_{(V_1)}$, and H_1, H_2 be subspaces of V_1. If $p = H_1$ and $q = H_2$, then $p \sqcap q = H_1 \cap H_2$.

Let L be a non empty lattice structure. Let us observe that L is complete if and only if the condition (Def. 6) is satisfied.

(Def. 6)   Let X be a subset of L. Then there exists an element a of L such that $a \sqsubseteq X$ and for every element b of L such that $b \sqsubseteq X$ holds $b \sqsubseteq a$.

Next we state two propositions:

(9)   For every V_1 holds $\mathbb{L}_{(V_1)}$ is complete.

(10)   Let x be a set, V_1 be a strict vector space over F, and S be a subset of V_1. If S is non empty and linearly closed, then if $x \in \mathrm{Lin}(S)$, then $x \in S$.

Let F be a field, let A, B be strict vector spaces over F, and let f be a function from the carrier of A into the carrier of B. The functor FuncLatt(f) yields a function from the carrier of $\mathbb{L}_A$ into the carrier of $\mathbb{L}_B$ and is defined as follows:

(Def. 7)   For every strict subspace S of A and for every subset B_0 of B such that $B_0 = f°$(the carrier of S) holds (FuncLatt(f))(S) = $\mathrm{Lin}(B_0)$.

Let L_1, L_2 be lattices. A function from the carrier of L_1 into the carrier of L_2 is said to be a lower homomorphism between L_1 and L_2 if:

(Def. 8)   For all elements a, b of L_1 holds it($a \sqcap b$) = it(a) $\sqcap$ it(b).

Let L_1, L_2 be lattices. A function from the carrier of L_1 into the carrier of L_2 is said to be an upper homomorphism between L_1 and L_2 if:

(Def. 9)   For all elements a, b of L_1 holds it($a \sqcup b$) = it(a) $\sqcup$ it(b).

The following propositions are true:

(11)   Let L_1, L_2 be lattices and f be a function from the carrier of L_1 into the carrier of L_2. Then f is a homomorphism from L_1 to L_2 if and only if f is an upper homomorphism between L_1 and L_2 and a lower homomorphism between L_1 and L_2.

(12)   Let F be a field, A, B be strict vector spaces over F, and f be a map from A into B. If f is linear, then FuncLatt(f) is an upper homomorphism between $\mathbb{L}_A$ and $\mathbb{L}_B$.

(13)   Let F be a field, A, B be strict vector spaces over F, and f be a map from A into B. Suppose f is one-to-one and linear. Then FuncLatt(f) is a homomorphism from $\mathbb{L}_A$ to $\mathbb{L}_B$.

(14)   Let A, B be strict vector spaces over F and f be a map from A into B. If f is linear and one-to-one, then FuncLatt(f) is one-to-one.

(15)   For every strict vector space A over F holds FuncLatt($\mathrm{id}_{\text{the carrier of } A}$) = $\mathrm{id}_{\text{the carrier of } \mathbb{L}_A}$.

REFERENCES

[1] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lattice3.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[4] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[5] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[6] Jolanta Kamieńska and Jarosław Stanisław Walijewski. Homomorphisms of lattices, finite join and finite meet. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/lattice4.html.

[7] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[8] Michał Muzalewski. Rings and modules — part II. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/mod_2.html.

[9] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[11] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[12] Wojciech A. Trybulec. Basis of vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_7.html.

[13] Wojciech A. Trybulec. Operations on subspaces in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_5.html.

[14] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_4.html.

[15] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[16] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[17] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/lattices.html.