JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Basis of Vector Space

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Summary. We prove the existence of a basis of a vector space, i.e., a set of vectors that generates the vector space and is linearly independent. We also introduce the notion of a subspace generated by a set of vectors and linear independence of set of vectors.

MML Identifier: VECTSP_7.

WWW: http://mizar.org/JFM/Vol2/vectsp_7.html

The articles [7], [4], [14], [15], [2], [3], [8], [5], [1], [9], [6], [10], [13], [12], and [11] provide the notation and terminology for this paper.

For simplicity, we use the following convention: x is a set, G_1 is a field, a, b are elements of G_1, V is a vector space over G_1, v, v_1, v_2 are vectors of V, A, B are subsets of V, and l is a linear combination of A.

Let us consider G_1, let us consider V, and let I_1 be a subset of V. We say that I_1 is linearly independent if and only if:

(Def. 1) For every linear combination l of I_1 such that $\sum l = 0_V$ holds the support of $l = \emptyset$.

We introduce I_1 is linearly dependent as an antonym of I_1 is linearly independent.

The following propositions are true:

(2)[1] If $A \subseteq B$ and B is linearly independent, then A is linearly independent.

(3) If A is linearly independent, then $0_V \notin A$.

(4) $\emptyset_{\text{the carrier of } V}$ is linearly independent.

(5) $\{v\}$ is linearly independent iff $v \neq 0_V$.

(6) If $\{v_1, v_2\}$ is linearly independent, then $v_1 \neq 0_V$ and $v_2 \neq 0_V$.

(7) $\{v, 0_V\}$ is linearly dependent and $\{0_V, v\}$ is linearly dependent.

(8) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent iff $v_2 \neq 0_V$ and for every a holds $v_1 \neq a \cdot v_2$.

(9) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent iff for all a, b such that $a \cdot v_1 + b \cdot v_2 = 0_V$ holds $a = 0_{(G_1)}$ and $b = 0_{(G_1)}$.

Let us consider G_1, let us consider V, and let us consider A. The functor $\mathrm{Lin}(A)$ yields a strict subspace of V and is defined as follows:

(Def. 2) The carrier of $\mathrm{Lin}(A) = \{\sum l\}$.

The following propositions are true:

[1] The proposition (1) has been removed.

(12)[2] $x \in \mathrm{Lin}(A)$ iff there exists l such that $x = \sum l$.

(13) If $x \in A$, then $x \in \mathrm{Lin}(A)$.

(14) $\mathrm{Lin}(\emptyset_{\text{the carrier of } V}) = \mathbf{0}_V$.

(15) If $\mathrm{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{0_V\}$.

(16) For every strict subspace W of V such that $A =$ the carrier of W holds $\mathrm{Lin}(A) = W$.

(17) For every strict vector space V over G_1 and for every subset A of V such that $A =$ the carrier of V holds $\mathrm{Lin}(A) = V$.

(18) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(19) For every strict vector space V over G_1 and for all subsets A, B of V such that $\mathrm{Lin}(A) = V$ and $A \subseteq B$ holds $\mathrm{Lin}(B) = V$.

(20) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(21) $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(22) Let V be a vector space over G_1 and A be a subset of V. Suppose A is linearly independent. Then there exists a subset B of V such that $A \subseteq B$ and B is linearly independent and $\mathrm{Lin}(B) =$ the vector space structure of V.

(23) If $\mathrm{Lin}(A) = V$, then there exists B such that $B \subseteq A$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

Let us consider G_1 and let V be a vector space over G_1. A subset of V is called a basis of V if:

(Def. 3) It is linearly independent and $\mathrm{Lin}(\mathrm{it}) =$ the vector space structure of V.

Next we state two propositions:

(27)[3] Let V be a vector space over G_1 and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

(28) For every vector space V over G_1 and for every subset A of V such that $\mathrm{Lin}(A) = V$ there exists a basis I of V such that $I \subseteq A$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[4] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[5] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[6] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[8] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fraenkel.html.

[2] The propositions (10) and (11) have been removed.
[3] The propositions (24)–(26) have been removed.

[9] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[10] Wojciech A. Trybulec. Linear combinations in real linear space. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rlvect_2.html`.

[11] Wojciech A. Trybulec. Linear combinations in vector space. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/vectsp_6.html`.

[12] Wojciech A. Trybulec. Operations on subspaces in vector space. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/vectsp_5.html`.

[13] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/vectsp_4.html`.

[14] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[15] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received July 27, 1990

Published January 2, 2004