JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

# Operations on Subspaces in Vector Space

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**Summary.** Sum, direct sum and intersection of subspaces are introduced. We prove some theorems concerning those notions and the decomposition of vector onto two subspaces. Linear complement of a subspace is also defined. We prove theorems that belong rather to [4].

MML Identifier: VECTSP_5.

WWW: http://mizar.org/JFM/Vol2/vectsp_5.html

The articles [6], [3], [9], [1], [10], [2], [12], [11], [7], [4], [5], and [8] provide the notation and terminology for this paper.

For simplicity, we use the following convention: $G_1$ denotes an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, $M$ denotes an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over $G_1$, $W$, $W_1$, $W_2$, $W_3$ denote subspaces of $M$, $u$, $v$, $v_1$, $v_2$ denote elements of $M$, and $x$ denotes a set.

Let us consider $G_1$, let us consider $M$, and let us consider $W_1$, $W_2$. The functor $W_1 + W_2$ yielding a strict subspace of $M$ is defined by:

(Def. 1) The carrier of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

Let us consider $G_1$, let us consider $M$, and let us consider $W_1$, $W_2$. The functor $W_1 \cap W_2$ yields a strict subspace of $M$ and is defined as follows:

(Def. 2) The carrier of $W_1 \cap W_2 =$ (the carrier of $W_1$) $\cap$ (the carrier of $W_2$).

Let us note that the functor $W_1 \cap W_2$ is commutative.

The following propositions are true:

(5)[1] $x \in W_1 + W_2$ iff there exist $v_1$, $v_2$ such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ iff $x \in W_1$ and $x \in W_2$.

(8) For every strict subspace $W$ of $M$ holds $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(11) $W_1$ is a subspace of $W_1 + W_2$ and $W_2$ is a subspace of $W_1 + W_2$.

(12) For every strict subspace $W_2$ of $M$ holds $W_1$ is a subspace of $W_2$ iff $W_1 + W_2 = W_2$.

[1] The propositions (1)–(4) have been removed.

(13) For every strict subspace $W$ of $M$ holds $\mathbf{0}_M + W = W$ and $W + \mathbf{0}_M = W$.

(14) $\mathbf{0}_M + \Omega_M =$ the vector space structure of $M$ and $\Omega_M + \mathbf{0}_M =$ the vector space structure of $M$.

(15) $\Omega_M + W =$ the vector space structure of $M$ and $W + \Omega_M =$ the vector space structure of $M$.

(16) Let $M$ be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over $G_1$. Then $\Omega_M + \Omega_M = M$.

(17) For every strict subspace $W$ of $M$ holds $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(20) $W_1 \cap W_2$ is a subspace of $W_1$ and $W_1 \cap W_2$ is a subspace of $W_2$.

(21)(i) For every strict subspace $W_1$ of $M$ such that $W_1$ is a subspace of $W_2$ holds $W_1 \cap W_2 = W_1$, and

(ii) for every $W_1$ such that $W_1 \cap W_2 = W_1$ holds $W_1$ is a subspace of $W_2$.

(22) If $W_1$ is a subspace of $W_2$, then $W_1 \cap W_3$ is a subspace of $W_2 \cap W_3$.

(23) If $W_1$ is a subspace of $W_3$, then $W_1 \cap W_2$ is a subspace of $W_3$.

(24) If $W_1$ is a subspace of $W_2$ and a subspace of $W_3$, then $W_1$ is a subspace of $W_2 \cap W_3$.

(25) $\mathbf{0}_M \cap W = \mathbf{0}_M$ and $W \cap \mathbf{0}_M = \mathbf{0}_M$.

(27)[2] For every strict subspace $W$ of $M$ holds $\Omega_M \cap W = W$ and $W \cap \Omega_M = W$.

(28) Let $M$ be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over $G_1$. Then $\Omega_M \cap \Omega_M = M$.

(29) $W_1 \cap W_2$ is a subspace of $W_1 + W_2$.

(30) For every strict subspace $W_2$ of $M$ holds $W_1 \cap W_2 + W_2 = W_2$.

(31) For every strict subspace $W_1$ of $M$ holds $W_1 \cap (W_1 + W_2) = W_1$.

(32) $W_1 \cap W_2 + W_2 \cap W_3$ is a subspace of $W_2 \cap (W_1 + W_3)$.

(33) If $W_1$ is a subspace of $W_2$, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3$ is a subspace of $(W_1 + W_2) \cap (W_2 + W_3)$.

(35) If $W_1$ is a subspace of $W_2$, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) For every strict subspace $W_1$ of $M$ such that $W_1$ is a subspace of $W_3$ holds $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(37) For all strict subspaces $W_1$, $W_2$ of $M$ holds $W_1 + W_2 = W_2$ iff $W_1 \cap W_2 = W_1$.

(38) For all strict subspaces $W_2$, $W_3$ of $M$ such that $W_1$ is a subspace of $W_2$ holds $W_1 + W_3$ is a subspace of $W_2 + W_3$.

(39) If $W_1$ is a subspace of $W_2$, then $W_1$ is a subspace of $W_2 + W_3$.

(40) If $W_1$ is a subspace of $W_3$ and $W_2$ is a subspace of $W_3$, then $W_1 + W_2$ is a subspace of $W_3$.

(41) There exists $W$ such that the carrier of $W =$ (the carrier of $W_1$) $\cup$ (the carrier of $W_2$) if and only if $W_1$ is a subspace of $W_2$ or $W_2$ is a subspace of $W_1$.

[2] The proposition (26) has been removed.

Let us consider $G_1$ and let us consider $M$. The functor Subspaces$M$ yields a set and is defined as follows:

(Def. 3) For every $x$ holds $x \in$ Subspaces$M$ iff there exists a strict subspace $W$ of $M$ such that $W = x$.

Let us consider $G_1$ and let us consider $M$. One can check that Subspaces$M$ is non empty.

We now state the proposition

(44)[3] Let $M$ be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over $G_1$. Then $M \in$ Subspaces$M$.

Let us consider $G_1$, let us consider $M$, and let us consider $W_1$, $W_2$. We say that $M$ is the direct sum of $W_1$ and $W_2$ if and only if:

(Def. 4) The vector space structure of $M = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_M$.

In the sequel $F$ is a field, $V$ is a vector space over $F$, and $W$ is a subspace of $V$.

Let us consider $F$, $V$, $W$. A subspace of $V$ is called a linear complement of $W$ if:

(Def. 5) $V$ is the direct sum of it and $W$.

In the sequel $W$, $W_1$, $W_2$ denote subspaces of $V$.

One can prove the following four propositions:

(47)[4] If $V$ is the direct sum of $W_1$ and $W_2$, then $W_2$ is a linear complement of $W_1$.

(48) For every linear complement $L$ of $W$ holds $V$ is the direct sum of $L$ and $W$ and the direct sum of $W$ and $L$.

(49) Let $L$ be a linear complement of $W$. Then $W + L =$ the vector space structure of $V$ and $L + W =$ the vector space structure of $V$.

(50) For every linear complement $L$ of $W$ holds $W \cap L = \mathbf{0}_V$ and $L \cap W = \mathbf{0}_V$.

In the sequel $W_1$, $W_2$ are subspaces of $M$.

The following two propositions are true:

(51) If $M$ is the direct sum of $W_1$ and $W_2$, then $M$ is the direct sum of $W_2$ and $W_1$.

(52) $M$ is the direct sum of $\mathbf{0}_M$ and $\Omega_M$ and the direct sum of $\Omega_M$ and $\mathbf{0}_M$.

In the sequel $W$ is a subspace of $V$.

Next we state two propositions:

(53) For every linear complement $L$ of $W$ holds $W$ is a linear complement of $L$.

(54) $\mathbf{0}_V$ is a linear complement of $\Omega_V$ and $\Omega_V$ is a linear complement of $\mathbf{0}_V$.

For simplicity, we adopt the following rules: $W_1$, $W_2$ denote subspaces of $M$, $v$ denotes an element of $M$, $C_1$ denotes a coset of $W_1$, and $C_2$ denotes a coset of $W_2$.

Next we state several propositions:

(55) If $C_1$ meets $C_2$, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(56) $M$ is the direct sum of $W_1$ and $W_2$ if and only if for every coset $C_1$ of $W_1$ and for every coset $C_2$ of $W_2$ there exists an element $v$ of $M$ such that $C_1 \cap C_2 = \{v\}$.

(57) Let $M$ be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over $G_1$ and $W_1$, $W_2$ be subspaces of $M$. Then $W_1 + W_2 = M$ if and only if for every element $v$ of $M$ there exist elements $v_1$, $v_2$ of $M$ such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

[3] The propositions (42) and (43) have been removed.

[4] The propositions (45) and (46) have been removed.

(58) Let $v$, $v_1$, $v_2$, $u_1$, $u_2$ be elements of $M$. Suppose $M$ is the direct sum of $W_1$ and $W_2$ and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$. Then $v_1 = u_1$ and $v_2 = u_2$.

(59) Suppose that

(i) $M = W_1 + W_2$, and

(ii) there exists $v$ such that for all elements $v_1$, $v_2$, $u_1$, $u_2$ of $M$ such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$.

Then $M$ is the direct sum of $W_1$ and $W_2$.

Let us consider $G_1$, $M$, $v$, $W_1$, $W_2$. Let us assume that $M$ is the direct sum of $W_1$ and $W_2$. The functor $v_{\langle W_1, W_2 \rangle}$ yields an element of $[:$the carrier of $M$, the carrier of $M:]$ and is defined as follows:

(Def. 6) $v = (v_{\langle W_1,W_2 \rangle})_{\mathbf{1}} + (v_{\langle W_1,W_2 \rangle})_{\mathbf{2}}$ and $(v_{\langle W_1,W_2 \rangle})_{\mathbf{1}} \in W_1$ and $(v_{\langle W_1,W_2 \rangle})_{\mathbf{2}} \in W_2$.

One can prove the following two propositions:

(64)[5] If $M$ is the direct sum of $W_1$ and $W_2$, then $(v_{\langle W_1,W_2 \rangle})_{\mathbf{1}} = (v_{\langle W_2,W_1 \rangle})_{\mathbf{2}}$.

(65) If $M$ is the direct sum of $W_1$ and $W_2$, then $(v_{\langle W_1,W_2 \rangle})_{\mathbf{2}} = (v_{\langle W_2,W_1 \rangle})_{\mathbf{1}}$.

In the sequel $W$ is a subspace of $V$.

One can prove the following propositions:

(66) Let $L$ be a linear complement of $W$, $v$ be an element of $V$, and $t$ be an element of $[:$the carrier of $V$, the carrier of $V:]$. If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v_{\langle W,L \rangle}$.

(67) For every linear complement $L$ of $W$ and for every element $v$ of $V$ holds $(v_{\langle W,L \rangle})_{\mathbf{1}} + (v_{\langle W,L \rangle})_{\mathbf{2}} = v$.

(68) For every linear complement $L$ of $W$ and for every element $v$ of $V$ holds $(v_{\langle W,L \rangle})_{\mathbf{1}} \in W$ and $(v_{\langle W,L \rangle})_{\mathbf{2}} \in L$.

(69) For every linear complement $L$ of $W$ and for every element $v$ of $V$ holds $(v_{\langle W,L \rangle})_{\mathbf{1}} = (v_{\langle L,W \rangle})_{\mathbf{2}}$.

(70) For every linear complement $L$ of $W$ and for every element $v$ of $V$ holds $(v_{\langle W,L \rangle})_{\mathbf{2}} = (v_{\langle L,W \rangle})_{\mathbf{1}}$.

In the sequel $A_1$, $A_2$ are elements of $\mathrm{Subspaces}\, M$ and $W_1$, $W_2$ are subspaces of $M$.

Let us consider $G_1$ and let us consider $M$. The functor $\mathrm{SubJoin}\, M$ yielding a binary operation on $\mathrm{Subspaces}\, M$ is defined as follows:

(Def. 7) For all $A_1$, $A_2$, $W_1$, $W_2$ such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\, M)(A_1, A_2) = W_1 + W_2$.

Let us consider $G_1$ and let us consider $M$. The functor $\mathrm{SubMeet}\, M$ yielding a binary operation on $\mathrm{Subspaces}\, M$ is defined as follows:

(Def. 8) For all $A_1$, $A_2$, $W_1$, $W_2$ such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\, M)(A_1, A_2) = W_1 \cap W_2$.

We now state several propositions:

(75)[6] $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is a lattice.

[5] The propositions (60)–(63) have been removed.
[6] The propositions (71)–(74) have been removed.

(76) $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is a lower bound lattice.

(77) $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is an upper bound lattice.

(78) $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is a bound lattice.

(79) $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is a modular lattice.

(80) For every field $F$ and for every vector space $V$ over $F$ holds $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a complemented lattice.

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*Received July 27, 1990*

*Published January 2, 2004*