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Construction of Rings and Left-, Right-, and Bi-Modules over a Ring

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Summary. Definitions of some classes of rings and left-, right-, and bi-modules over a ring and some elementary theorems on rings and skew fields.

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The articles [7], [4], [9], [3], [2], [8], [5], [6], and [1] provide the notation and terminology for this paper.

In this paper F_1 denotes a non empty double loop structure and F denotes a field.

Let I_1 be a non empty multiplicative loop structure. We say that I_1 is well unital if and only if:

(Def. 2)[1] For every element x of I_1 holds $x \cdot \mathbf{1}_{(I_1)} = x$ and $\mathbf{1}_{(I_1)} \cdot x = x$.

One can verify the following observations:

* every non empty multiplicative loop structure which is well unital is also left unital and right unital,
* every non empty multiplicative loop structure which is left unital and right unital is also well unital, and
* there exists a non empty double loop structure which is strict, Abelian, add-associative, right zeroed, right complementable, well unital, and distributive.

The following proposition is true

(1) For all scalars x, y, z of F_1 holds $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x+0_{(F_1)} = x$ and $x+-x = 0_{(F_1)}$ and $x \cdot \mathbf{1}_{(F_1)} = x$ and $\mathbf{1}_{(F_1)} \cdot x = x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(y+z) \cdot x = y \cdot x + z \cdot x$ if and only if F_1 is a ring.

Let us note that there exists a ring which is strict.

Let us note that there exists a ring which is commutative.

A commutative ring is a commutative ring.

One can check that there exists a commutative ring which is strict.

Let I_1 be a non empty multiplicative loop with zero structure. We say that I_1 is integral domain-like if and only if:

(Def. 5)[2] For all elements x, y of I_1 such that $x \cdot y = 0_{(I_1)}$ holds $x = 0_{(I_1)}$ or $y = 0_{(I_1)}$.

[1] The definition (Def. 1) has been removed.
[2] The definitions (Def. 3) and (Def. 4) have been removed.

One can check that there exists a commutative ring which is strict, non degenerated, and integral domain-like.

An integral domain is an integral domain-like non degenerated commutative ring.

Next we state the proposition

(13)[3] F is an integral domain.

Let us mention that there exists a ring which is non degenerated and field-like.

A skew field is a non degenerated field-like ring.

Let us mention that there exists a skew field which is strict.

In the sequel R is a ring.

The following propositions are true:

(16)[4] Suppose that for every scalar x of R holds if $x \neq 0_R$, then there exists a scalar y of R such that $x \cdot y = \mathbf{1}_R$ and $0_R \neq \mathbf{1}_R$. Then R is a skew field.

(19)[5] F is a skew field.

Let us note that every non empty multiplicative loop structure which is commutative and left unital is also well unital and every non empty multiplicative loop structure which is commutative and right unital is also well unital.

In the sequel R denotes an Abelian add-associative right zeroed right complementable non empty loop structure and x, y, z denote scalars of R.

The following three propositions are true:

(22)[6] $x + y = z$ iff $x = z - y$ and $x + y = z$ iff $y = z - x$.

(34)[7] Let R be an add-associative right zeroed right complementable non empty loop structure and x be an element of R. Then $x = 0_R$ if and only if $-x = 0_R$.

(38)[8] Let R be an add-associative right zeroed Abelian right complementable non empty loop structure and x, y be elements of R. Then there exists an element z of R such that $x = y + z$ and $x = z + y$.

In the sequel S_1 denotes a skew field and x, y, z denote scalars of S_1.

The following propositions are true:

(39) Let F be an add-associative right zeroed right complementable distributive non degenerated non empty double loop structure and x, y be elements of F. If $x \cdot y = \mathbf{1}_F$, then $x \neq 0_F$ and $y \neq 0_F$.

(40) Let S_1 be a non degenerated field-like associative Abelian add-associative right zeroed right complementable well unital distributive non empty double loop structure and x be an element of S_1. If $x \neq 0_{(S_1)}$, then there exists an element y of S_1 such that $y \cdot x = \mathbf{1}_{(S_1)}$.

(41) If $x \cdot y = \mathbf{1}_{(S_1)}$, then $y \cdot x = \mathbf{1}_{(S_1)}$.

(42) Let S_1 be a non degenerated field-like associative Abelian add-associative right zeroed right complementable well unital distributive non empty double loop structure and x, y, z be elements of S_1. If $x \cdot y = x \cdot z$ and $x \neq 0_{(S_1)}$, then $y = z$.

Let S_1 be a non degenerated field-like associative Abelian add-associative right zeroed right complementable well unital distributive non empty double loop structure and let x be an element of S_1. Let us assume that $x \neq 0_{(S_1)}$. The functor x^{-1} yielding a scalar of S_1 is defined as follows:

[3] The propositions (2)–(12) have been removed.
[4] The propositions (14) and (15) have been removed.
[5] The propositions (17) and (18) have been removed.
[6] The propositions (20) and (21) have been removed.
[7] The propositions (23)–(33) have been removed.
[8] The propositions (35)–(37) have been removed.

(Def. 7)[9] $x \cdot x^{-1} = \mathbf{1}_{(S_1)}$.

Let us consider S_1, x, y. The functor $\frac{x}{y}$ yielding a scalar of S_1 is defined by:

(Def. 8) $\frac{x}{y} = x \cdot y^{-1}$.

One can prove the following propositions:

(43) If $x \neq 0_{(S_1)}$, then $x \cdot x^{-1} = \mathbf{1}_{(S_1)}$ and $x^{-1} \cdot x = \mathbf{1}_{(S_1)}$.

(45)[10] If $x \cdot y = \mathbf{1}_{(S_1)}$, then $x = y^{-1}$ and $y = x^{-1}$.

(46) If $x \neq 0_{(S_1)}$ and $y \neq 0_{(S_1)}$, then $x^{-1} \cdot y^{-1} = (y \cdot x)^{-1}$.

(47) If $x \cdot y = 0_{(S_1)}$, then $x = 0_{(S_1)}$ or $y = 0_{(S_1)}$.

(48) If $x \neq 0_{(S_1)}$, then $x^{-1} \neq 0_{(S_1)}$.

(49) If $x \neq 0_{(S_1)}$, then $(x^{-1})^{-1} = x$.

(50) If $x \neq 0_{(S_1)}$, then $\frac{\mathbf{1}_{(S_1)}}{x} = x^{-1}$ and $\frac{\mathbf{1}_{(S_1)}}{x^{-1}} = x$.

(51) If $x \neq 0_{(S_1)}$, then $x \cdot \frac{\mathbf{1}_{(S_1)}}{x} = \mathbf{1}_{(S_1)}$ and $\frac{\mathbf{1}_{(S_1)}}{x} \cdot x = \mathbf{1}_{(S_1)}$.

(52) If $x \neq 0_{(S_1)}$, then $\frac{x}{x} = \mathbf{1}_{(S_1)}$.

(53) If $y \neq 0_{(S_1)}$ and $z \neq 0_{(S_1)}$, then $\frac{x}{y} = \frac{x \cdot z}{y \cdot z}$.

(54) If $y \neq 0_{(S_1)}$, then $-\frac{x}{y} = \frac{-x}{y}$ and $\frac{x}{-y} = -\frac{x}{y}$.

(55) If $z \neq 0_{(S_1)}$, then $\frac{x}{z} + \frac{y}{z} = \frac{x+y}{z}$ and $\frac{x}{z} - \frac{y}{z} = \frac{x-y}{z}$.

(56) If $y \neq 0_{(S_1)}$ and $z \neq 0_{(S_1)}$, then $\frac{x}{\frac{y}{z}} = \frac{x \cdot z}{y}$.

(57) If $y \neq 0_{(S_1)}$, then $\frac{x}{y} \cdot y = x$.

Let F_1 be a 1-sorted structure. We consider right module structures over F_1 as extensions of loop structure as systems

⟨ a carrier, an addition, a zero, a right multiplication ⟩,

where the carrier is a set, the addition is a binary operation on the carrier, the zero is an element of the carrier, and the right multiplication is a function from [: the carrier, the carrier of F_1 :] into the carrier.

Let F_1 be a 1-sorted structure. Observe that there exists a right module structure over F_1 which is non empty.

Let F_1 be a 1-sorted structure, let A be a non empty set, let a be a binary operation on A, let Z be an element of A, and let r be a function from [: A, the carrier of F_1 :] into A. Note that $\langle A, a, Z, r \rangle$ is non empty.

Let us consider F_1 and let R_1 be a non empty right module structure over F_1. A scalar of R_1 is an element of F_1. A vector of R_1 is an element of R_1.

Let F_2, F_3 be 1-sorted structures. We introduce bimodule structures over F_2, F_3 which are extensions of vector space structure over F_2 and right module structure over F_3 and are systems

⟨ a carrier, an addition, a zero, a left multiplication, a right multiplication ⟩,

where the carrier is a set, the addition is a binary operation on the carrier, the zero is an element of the carrier, the left multiplication is a function from [: the carrier of F_2, the carrier :] into the carrier, and the right multiplication is a function from [: the carrier, the carrier of F_3 :] into the carrier.

Let F_2, F_3 be 1-sorted structures. Observe that there exists a bimodule structure over F_2, F_3 which is non empty.

[9] The definition (Def. 6) has been removed.

[10] The proposition (44) has been removed.

Let F_2, F_3 be 1-sorted structures, let A be a non empty set, let a be a binary operation on A, let Z be an element of A, let l be a function from $[:$the carrier of F_2, $A:]$ into A, and let r be a function from $[:A$, the carrier of $F_3:]$ into A. Note that $\langle A, a, Z, l, r \rangle$ is non empty.

In the sequel R, R_2, R_3 are rings.

Let R be an Abelian add-associative right zeroed right complementable non empty loop structure. The functor $\mathrm{AbGr}(R)$ yielding a strict Abelian group is defined as follows:

(Def. 9) $\mathrm{AbGr}(R) = \langle$the carrier of R, the addition of R, the zero of $R\rangle$.

Let us consider R. Note that there exists a non empty vector space structure over R which is Abelian, add-associative, right zeroed, right complementable, and strict.

Let us consider R. The functor $\mathrm{LeftMod}(R)$ yielding an Abelian add-associative right zeroed right complementable strict non empty vector space structure over R is defined as follows:

(Def. 11)[11] $\mathrm{LeftMod}(R) = \langle$the carrier of R, the addition of R, the zero of R, the multiplication of $R\rangle$.

Let us consider R. Note that there exists a non empty right module structure over R which is Abelian, add-associative, right zeroed, right complementable, and strict.

Let us consider R. The functor $\mathrm{RightMod}(R)$ yields an Abelian add-associative right zeroed right complementable strict non empty right module structure over R and is defined by:

(Def. 14)[12] $\mathrm{RightMod}(R) = \langle$the carrier of R, the addition of R, the zero of R, the multiplication of $R\rangle$.

Let R be a non empty 1-sorted structure, let V be a non empty right module structure over R, let x be an element of R, and let v be an element of V. The functor $v \cdot x$ yields an element of V and is defined as follows:

(Def. 15) $v \cdot x =$ (the right multiplication of $V)(v, x)$.

(Def. 17) op_1 is a unary operation on $\{\emptyset\}$.

(Def. 18) op_0 is an element of $\{\emptyset\}$.

Let us consider R_2, R_3. Note that there exists a non empty bimodule structure over R_2, R_3 which is Abelian, add-associative, right zeroed, right complementable, and strict.

Let us consider R_2, R_3. The functor $\mathrm{BiMod}(R_2, R_3)$ yields an Abelian add-associative right zeroed right complementable strict non empty bimodule structure over R_2, R_3 and is defined by:

(Def. 21)[13] $\mathrm{BiMod}(R_2, R_3) = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_0, \pi_2((\text{the carrier of } R_2) \times \{\emptyset\}), \pi_1(\{\emptyset\} \times \text{the carrier of } R_3)\rangle$.

One can prove the following proposition

(71)[14] Let x, y be scalars of R and v, w be vectors of $\mathrm{LeftMod}(R)$. Then $x \cdot (v + w) = x \cdot v + x \cdot w$ and $(x + y) \cdot v = x \cdot v + y \cdot v$ and $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ and $\mathbf{1}_R \cdot v = v$.

Let us consider R. One can check that there exists a non empty vector space structure over R which is vector space-like, Abelian, add-associative, right zeroed, right complementable, and strict.

Let us consider R. A left module over R is an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over R.

Let us consider R. Observe that $\mathrm{LeftMod}(R)$ is Abelian, add-associative, right zeroed, right complementable, strict, and vector space-like.

One can prove the following proposition

(77)[15] Let x, y be scalars of R and v, w be vectors of $\mathrm{RightMod}(R)$. Then $(v + w) \cdot x = v \cdot x + w \cdot x$ and $v \cdot (x + y) = v \cdot x + v \cdot y$ and $v \cdot (y \cdot x) = (v \cdot y) \cdot x$ and $v \cdot \mathbf{1}_R = v$.

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[11] The definition (Def. 10) has been removed.
[12] The definitions (Def. 12) and (Def. 13) have been removed.
[13] The definitions (Def. 18)–(Def. 20) have been removed.
[14] The propositions (58)–(70) have been removed.
[15] The propositions (72)–(76) have been removed.

Let R be a non empty double loop structure and let I_1 be a non empty right module structure over R. We say that I_1 is right module-like if and only if the condition (Def. 23) is satisfied.

(Def. 23)[16] Let x, y be scalars of R and v, w be vectors of I_1. Then $(v+w)\cdot x = v\cdot x + w\cdot x$ and $v\cdot(x+y) = v\cdot x + v\cdot y$ and $v\cdot(y\cdot x) = (v\cdot y)\cdot x$ and $v\cdot \mathbf{1}_R = v$.

Let us consider R. Observe that there exists a non empty right module structure over R which is Abelian, add-associative, right zeroed, right complementable, right module-like, and strict.

Let us consider R. A right module over R is an Abelian add-associative right zeroed right complementable right module-like non empty right module structure over R.

Let us consider R. One can verify that RightMod(R) is Abelian, add-associative, right zeroed, right complementable, and right module-like.

Let us consider R_2, R_3 and let I_1 be a non empty bimodule structure over R_2, R_3. We say that I_1 is bimodule-like if and only if:

(Def. 24) For every scalar x of R_2 and for every scalar p of R_3 and for every vector v of I_1 holds $x\cdot(v\cdot p) = (x\cdot v)\cdot p$.

Let us consider R_2, R_3. Note that there exists a non empty bimodule structure over R_2, R_3 which is Abelian, add-associative, right zeroed, right complementable, right module-like, vector space-like, bimodule-like, and strict.

Let us consider R_2, R_3. A bimodule over R_2 and R_3 is an Abelian add-associative right zeroed right complementable right module-like vector space-like bimodule-like non empty bimodule structure over R_2, R_3.

Next we state two propositions:

(83)[17] Let V be a non empty bimodule structure over R_2, R_3. Then the following statements are equivalent

(i) for all scalars x, y of R_2 and for all scalars p, q of R_3 and for all vectors v, w of V holds $x\cdot(v+w) = x\cdot v + x\cdot w$ and $(x+y)\cdot v = x\cdot v + y\cdot v$ and $(x\cdot y)\cdot v = x\cdot(y\cdot v)$ and $\mathbf{1}_{(R_2)}\cdot v = v$ and $(v+w)\cdot p = v\cdot p + w\cdot p$ and $v\cdot(p+q) = v\cdot p + v\cdot q$ and $v\cdot(q\cdot p) = (v\cdot q)\cdot p$ and $v\cdot\mathbf{1}_{(R_3)} = v$ and $x\cdot(v\cdot p) = (x\cdot v)\cdot p$,

(ii) V is right module-like, vector space-like, and bimodule-like.

(84) BiMod(R_2,R_3) is a bimodule over R_2 and R_3.

Let us consider R_2, R_3. One can verify that BiMod(R_2,R_3) is Abelian, add-associative, right zeroed, right complementable, right module-like, vector space-like, and bimodule-like.

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