JOURNAL OF FORMALIZED MATHEMATICS
Volume 13, Released 2001, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Urysohn Lemma

Józef Białas
Łódź University

Yatsuka Nakamura
Shinshu University
Nagano

Summary. This article is the third part of a paper proving the fundamental Urysohn Theorem concerning the existence of a real valued continuous function on a normal topological space. The paper is divided into two parts. In the first part, we describe the construction of the function solving thesis of the Urysohn Lemma. The second part contains the proof of the Urysohn Lemma in normal space and the proof of the same theorem for compact space.

MML Identifier: URYSOHN3.

WWW: http://mizar.org/JFM/Vol13/urysohn3.html

The articles [17], [19], [2], [18], [1], [20], [8], [9], [14], [12], [11], [15], [13], [7], [16], [10], [3], [4], [5], and [6] provide the notation and terminology for this paper.

Let D be a non empty subset of $\mathbb{R}$. Observe that every element of D is real.

One can prove the following proposition

(1) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let n be a natural number. Then there exists a function G from $\mathrm{dyadic}(n)$ into $2^{\text{the carrier of } T}$ such that $A \subseteq G(0)$ and $B = \Omega_T \setminus G(1)$ and for all elements r_1, r_2 of $\mathrm{dyadic}(n)$ such that $r_1 < r_2$ holds $G(r_1)$ is open and $G(r_2)$ is open and $\overline{G(r_1)} \subseteq G(r_2)$.

Let T be a non empty topological space, let A, B be subsets of T, and let n be a natural number. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and A misses B. A function from $\mathrm{dyadic}(n)$ into $2^{\text{the carrier of } T}$ is said to be a drizzle of A, B, n if it satisfies the conditions (Def. 1).

(Def. 1)(i) $A \subseteq \mathrm{it}(0)$,

(ii) $B = \Omega_T \setminus \mathrm{it}(1)$, and

(iii) for all elements r_1, r_2 of $\mathrm{dyadic}(n)$ such that $r_1 < r_2$ holds $\mathrm{it}(r_1)$ is open and $\mathrm{it}(r_2)$ is open and $\overline{\mathrm{it}(r_1)} \subseteq \mathrm{it}(r_2)$.

Next we state the proposition

(3)[1] Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let n be a natural number and G be a drizzle of A, B, n. Then there exists a drizzle F of A, B, $n+1$ such that for every element r of $\mathrm{dyadic}(n+1)$ if $r \in \mathrm{dyadic}(n)$, then $F(r) = G(r)$.

Let A, B be non empty sets, let F be a function from $\mathbb{N}$ into $A \dot{\rightarrow} B$, and let n be a natural number. Then $F(n)$ is a partial function from A to B.

One can prove the following proposition

[1] The proposition (2) has been removed.

(4) Let T be a non empty topological space, A, B be subsets of T, and n be a natural number. Then every drizzle of A, B, n is an element of DYADIC $\dot{\to} 2^{\text{the carrier of } T}$.

Let A, B be non empty sets, let F be a function from $\mathbb{N}$ into $A \dot{\to} B$, and let n be a natural number. Then $F(n)$ is an element of $A \dot{\to} B$.

We now state the proposition

(5) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Then there exists a sequence F of partial functions from DYADIC into $2^{\text{the carrier of } T}$ such that for every natural number n holds

$F(n)$ is a drizzle of A, B, n and for every element r of $\operatorname{dom} F(n)$ holds $F(n)(r) = F(n+1)(r)$.

Let T be a non empty topological space and let A, B be subsets of T. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and A misses B. A sequence of partial functions from DYADIC into $2^{\text{the carrier of } T}$ is said to be a rain of A, B if it satisfies the condition (Def. 2).

(Def. 2) Let n be a natural number. Then $\mathrm{it}(n)$ is a drizzle of A, B, n and for every element r of $\operatorname{dom} \mathrm{it}(n)$ holds $\mathrm{it}(n)(r) = \mathrm{it}(n+1)(r)$.

Let x be a real number. Let us assume that $x \in$ DYADIC. The functor InfDyadic x yielding a natural number is defined as follows:

(Def. 3) $x \in \mathrm{dyadic}(0)$ iff $\mathrm{InfDyadic}\, x = 0$ and for every natural number n such that $x \in \mathrm{dyadic}(n+1)$ and $x \notin \mathrm{dyadic}(n)$ holds $\mathrm{InfDyadic}\, x = n+1$.

We now state several propositions:

(6) For every real number x such that $x \in$ DYADIC holds $x \in \mathrm{dyadic}(\mathrm{InfDyadic}\, x)$.

(7) For every real number x such that $x \in$ DYADIC and for every natural number n such that $\mathrm{InfDyadic}\, x \leq n$ holds $x \in \mathrm{dyadic}(n)$.

(8) For every real number x and for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $\mathrm{InfDyadic}\, x \leq n$.

(9) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B and x be a real number. If $x \in$ DYADIC, then for every natural number n holds $G(\mathrm{InfDyadic}\, x)(x) = G(\mathrm{InfDyadic}\, x + n)(x)$.

(10) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B and x be a real number. Suppose $x \in$ DYADIC. Then there exists an element y of $2^{\text{the carrier of } T}$ such that for every natural number n if $x \in \mathrm{dyadic}(n)$, then $y = G(n)(x)$.

(11) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B. Then there exists a function F from DOM into $2^{\text{the carrier of } T}$ such that for every real number x holds

(i) if $x \in \mathbb{R}_{<0}$, then $F(x) = \emptyset$,

(ii) if $x \in \mathbb{R}_{>1}$, then $F(x) =$ the carrier of T, and

(iii) if $x \in$ DYADIC, then for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $F(x) = G(n)(x)$.

Let T be a non empty topological space and let A, B be subsets of T. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and A misses B. Let R be a rain of A, B. The functor Tempest R yielding a function from DOM into $2^{\text{the carrier of } T}$ is defined by the condition (Def. 4).

(Def. 4) Let x be a real number such that $x \in$ DOM. Then

(i) if $x \in \mathbb{R}_{<0}$, then $(\mathrm{Tempest}\, R)(x) = \emptyset$,

(ii) if $x \in \mathbb{R}_{>1}$, then $(\mathrm{Tempest}\, R)(x) =$ the carrier of T, and

(iii) if $x \in$ DYADIC, then for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $(\mathrm{Tempest}\, R)(x) = R(n)(x)$.

Let X be a non empty set, let T be a topological space, let F be a function from X into $2^{\text{the carrier of } T}$, and let x be an element of X. Then $F(x)$ is a subset of T.

One can prove the following three propositions:

(12) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B, r be a real number, and C be a subset of T. If $C = (\mathrm{Tempest}\, G)(r)$ and $r \in \mathrm{DOM}$, then C is open.

(13) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B and r_1, r_2 be real numbers. Suppose $r_1 \in \mathrm{DOM}$ and $r_2 \in \mathrm{DOM}$ and $r_1 < r_2$. Let C be a subset of T. If $C = (\mathrm{Tempest}\, G)(r_1)$, then $\overline{C} \subseteq (\mathrm{Tempest}\, G)(r_2)$.

(14) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, and p be a point of T. Then there exists a subset R of $\overline{\mathbb{R}}$ such that for every set x holds $x \in R$ if and only if the following conditions are satisfied:

(i) $x \in \mathrm{DYADIC}$, and

(ii) for every real number s such that $s = x$ holds $p \notin (\mathrm{Tempest}\, G)(s)$.

Let T be a non empty topological space, let A, B be subsets of T, let R be a rain of A, B, and let p be a point of T. The functor $\mathrm{Rainbow}(p,R)$ yielding a subset of $\overline{\mathbb{R}}$ is defined by:

(Def. 5) For every set x holds $x \in \mathrm{Rainbow}(p,R)$ iff $x \in \mathrm{DYADIC}$ and for every real number s such that $s = x$ holds $p \notin (\mathrm{Tempest}\, R)(s)$.

Let T, S be non empty topological spaces, let F be a function from the carrier of T into the carrier of S, and let p be a point of T. Then $F(p)$ is a point of S.

We now state two propositions:

(15) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, and p be a point of T. Then $\mathrm{Rainbow}(p,G) \subseteq \mathrm{DYADIC}$.

(16) Let T be a non empty topological space, A, B be subsets of T, and R be a rain of A, B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of T holds

if $\mathrm{Rainbow}(p,R) = \emptyset$, then $F(p) = 0$ and for every non empty subset S of $\overline{\mathbb{R}}$ such that $S = \mathrm{Rainbow}(p,R)$ holds $F(p) = \sup S$.

Let T be a non empty topological space, let A, B be subsets of T, and let R be a rain of A, B. The functor $\mathrm{Thunder}\, R$ yielding a map from T into $\mathbb{R}^{\mathbf{1}}$ is defined by the condition (Def. 6).

(Def. 6) Let p be a point of T. Then if $\mathrm{Rainbow}(p,R) = \emptyset$, then $(\mathrm{Thunder}\, R)(p) = 0$ and for every non empty subset S of $\overline{\mathbb{R}}$ such that $S = \mathrm{Rainbow}(p,R)$ holds $(\mathrm{Thunder}\, R)(p) = \sup S$.

Let T be a non empty topological space, let F be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let p be a point of T. Then $F(p)$ is a real number.

The following propositions are true:

(17) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, p be a point of T, and S be a non empty subset of $\overline{\mathbb{R}}$. Suppose $S = \mathrm{Rainbow}(p,G)$. Let ℓ_1 be an extended real number. If $\ell_1 = 1$, then $0_{\overline{\mathbb{R}}} \leq \sup S$ and $\sup S \leq \ell_1$.

(18) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B, r be an element of DOM, and p be a point of T. If $(\mathrm{Thunder}\, G)(p) < r$, then $p \in (\mathrm{Tempest}\, G)(r)$.

(19) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B and r be a real number. Suppose $r \in \mathrm{DYADIC} \cup \mathbb{R}_{>1}$ and $0 < r$. Let p be a point of T. If $p \in (\mathrm{Tempest}\, G)(r)$, then $(\mathrm{Thunder}\, G)(p) \leq r$.

(20) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B, n be a natural number, and r_1 be an element of DOM. If $0 < r_1$, then for every point p of T such that $r_1 < (\mathrm{Thunder}\,G)(p)$ holds $p \notin (\mathrm{Tempest}\,G)(r_1)$.

(21) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B. Then

(i) Thunder G is continuous, and

(ii) for every point x of T holds $0 \leq (\mathrm{Thunder}\,G)(x)$ and $(\mathrm{Thunder}\,G)(x) \leq 1$ and if $x \in A$, then $(\mathrm{Thunder}\,G)(x) = 0$ and if $x \in B$, then $(\mathrm{Thunder}\,G)(x) = 1$.

(22) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leq F(x)$ and $F(x) \leq 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

(23) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose A misses B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leq F(x)$ and $F(x) \leq 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

(24) Let T be a non empty T_2 compact topological space and A, B be closed subsets of T. Suppose A misses B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leq F(x)$ and $F(x) \leq 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[3] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/supinf_1.html`.

[4] Józef Białas. Series of positive real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/supinf_2.html`.

[5] Józef Białas. Properties of the intervals of real numbers. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/measure5.html`.

[6] Józef Białas and Yatsuka Nakamura. Dyadic numbers and T_4 topological spaces. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/urysohn1.html`.

[7] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/pcomps_1.html`.

[8] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[9] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[10] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[11] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/compts_1.html`.

[12] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces — fundamental concepts. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/topmetr.html`.

[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/metric_1.html`.

[14] Rafał Kwiatek. Factorial and Newton coefficients. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/newton.html.

[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[16] Beata Perkowska. Functional sequence from a domain to a domain. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/seqfunc.html.

[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[18] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[19] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[20] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received February 16, 2001

Published January 2, 2004