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Dyadic Numbers and T_4 Topological Spaces

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Summary. This article is the first part of a paper proving the fundamental Urysohn's Theorem concerning the existence of a real valued continuous function on a normal topological space. The paper is divided into four parts. In the first part, we prove some auxiliary theorems concerning properties of natural numbers and prove two useful schemes about recurrently defined functions; in the second part, we define a special set of rational numbers, which we call dyadic, and prove some of its properties. The next part of the paper contains the definitions of T_1 space and normal space, and we prove related theorems used in later parts of the paper. The final part of this work is developed for proving the theorem about the existence of some special family of subsets of a topological space. This theorem is essential in proving Urysohn's Lemma.

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The articles [13], [6], [15], [14], [12], [4], [5], [9], [3], [1], [8], [7], [11], [2], and [10] provide the notation and terminology for this paper.

1. Dyadic Numbers

The subset $\mathbb{R}_{<0}$ of $\mathbb{R}$ is defined as follows:

(Def. 1) For every real number x holds $x \in \mathbb{R}_{<0}$ iff $x < 0$.

The subset $\mathbb{R}_{>1}$ of $\mathbb{R}$ is defined by:

(Def. 2) For every real number x holds $x \in \mathbb{R}_{>1}$ iff $1 < x$.

Let n be a natural number. The functor $\text{dyadic}(n)$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 3) For every real number x holds $x \in \text{dyadic}(n)$ iff there exists a natural number i such that $0 \leq i$ and $i \leq 2^n$ and $x = \frac{i}{2^n}$.

The subset DYADIC of $\mathbb{R}$ is defined as follows:

(Def. 4) For every real number a holds $a \in \text{DYADIC}$ iff there exists a natural number n such that $a \in \text{dyadic}(n)$.

The subset DOM of $\mathbb{R}$ is defined as follows:

(Def. 5) $\text{DOM} = \mathbb{R}_{<0} \cup \text{DYADIC} \cup \mathbb{R}_{>1}$.

Let T be a topological space, let A be a non empty subset of $\mathbb{R}$, let F be a function from A into $2^{\text{the carrier of } T}$, and let r be an element of A. Then $F(r)$ is a subset of T.

Next we state three propositions:

(5)[1] For every natural number n and for every real number x such that $x \in \text{dyadic}(n)$ holds $0 \leq x$

[1] The propositions (1)–(4) have been removed.

and $x \leq 1$.

(6) $\mathrm{dyadic}(0) = \{0, 1\}$.

(7) $\mathrm{dyadic}(1) = \{0, \frac{1}{2}, 1\}$.

Let n be a natural number. Observe that $\mathrm{dyadic}(n)$ is non empty.
We now state two propositions:

(8) For all natural numbers x, n holds x^n is a natural number.

(9) Let n be a natural number. Then there exists a finite sequence F_1 such that $\mathrm{dom}\,F_1 = \mathrm{Seg}(2^n + 1)$ and for every natural number i such that $i \in \mathrm{dom}\,F_1$ holds $F_1(i) = \frac{i-1}{2^n}$.

Let n be a natural number. The functor $\mathrm{dyad}(n)$ yielding a finite sequence is defined by:

(Def. 6) $\mathrm{dom}\,\mathrm{dyad}(n) = \mathrm{Seg}(2^n + 1)$ and for every natural number i such that $i \in \mathrm{dom}\,\mathrm{dyad}(n)$ holds $(\mathrm{dyad}(n))(i) = \frac{i-1}{2^n}$.

We now state the proposition

(10) For every natural number n holds $\mathrm{dom}\,\mathrm{dyad}(n) = \mathrm{Seg}(2^n + 1)$ and $\mathrm{rng}\,\mathrm{dyad}(n) = \mathrm{dyadic}(n)$.

Let us observe that DYADIC is non empty.
Let us mention that DOM is non empty.
The following propositions are true:

(11) For every natural number n holds $\mathrm{dyadic}(n) \subseteq \mathrm{dyadic}(n+1)$.

(12) For every natural number n holds $0 \in \mathrm{dyadic}(n)$ and $1 \in \mathrm{dyadic}(n)$.

(13) For all natural numbers n, i such that $0 < i$ and $i \leq 2^n$ holds $\frac{i \cdot 2 - 1}{2^{n+1}} \in \mathrm{dyadic}(n+1) \setminus \mathrm{dyadic}(n)$.

(14) For all natural numbers n, i such that $0 \leq i$ and $i < 2^n$ holds $\frac{i \cdot 2 + 1}{2^{n+1}} \in \mathrm{dyadic}(n+1) \setminus \mathrm{dyadic}(n)$.

(15) For every natural number n holds $\frac{1}{2^{n+1}} \in \mathrm{dyadic}(n+1) \setminus \mathrm{dyadic}(n)$.

Let n be a natural number and let x be an element of $\mathrm{dyadic}(n)$. The functor $\mathrm{axis}(x, n)$ yields a natural number and is defined as follows:

(Def. 7) $x = \frac{\mathrm{axis}(x,n)}{2^n}$.

Next we state several propositions:

(16) For every natural number n and for every element x of $\mathrm{dyadic}(n)$ holds $x = \frac{\mathrm{axis}(x,n)}{2^n}$ and $0 \leq \mathrm{axis}(x, n)$ and $\mathrm{axis}(x, n) \leq 2^n$.

(17) For every natural number n and for every element x of $\mathrm{dyadic}(n)$ holds $\frac{\mathrm{axis}(x,n)-1}{2^n} < x$ and $x < \frac{\mathrm{axis}(x,n)+1}{2^n}$.

(18) For every natural number n and for every element x of $\mathrm{dyadic}(n)$ holds $\frac{\mathrm{axis}(x,n)-1}{2^n} < \frac{\mathrm{axis}(x,n)+1}{2^n}$.

(20)[2] Let n be a natural number and x be an element of $\mathrm{dyadic}(n+1)$. If $x \notin \mathrm{dyadic}(n)$, then $\frac{\mathrm{axis}(x,n+1)-1}{2^{n+1}} \in \mathrm{dyadic}(n)$ and $\frac{\mathrm{axis}(x,n+1)+1}{2^{n+1}} \in \mathrm{dyadic}(n)$.

(21) For every natural number n and for all elements x_1, x_2 of $\mathrm{dyadic}(n)$ such that $x_1 < x_2$ holds $\mathrm{axis}(x_1, n) < \mathrm{axis}(x_2, n)$.

(22) For every natural number n and for all elements x_1, x_2 of $\mathrm{dyadic}(n)$ such that $x_1 < x_2$ holds $x_1 \leq \frac{\mathrm{axis}(x_2,n)-1}{2^n}$ and $\frac{\mathrm{axis}(x_1,n)+1}{2^n} \leq x_2$.

(23) Let n be a natural number and x_1, x_2 be elements of $\mathrm{dyadic}(n+1)$. If $x_1 < x_2$ and $x_1 \notin \mathrm{dyadic}(n)$ and $x_2 \notin \mathrm{dyadic}(n)$, then $\frac{\mathrm{axis}(x_1,n+1)+1}{2^{n+1}} \leq \frac{\mathrm{axis}(x_2,n+1)-1}{2^{n+1}}$.

[2] The proposition (19) has been removed.

2. Normal Spaces

Let T be a non empty topological space and let x be a point of T. Let us note that the neighbourhood of x can be characterized by the following (equivalent) condition:

(Def. 8) There exists a subset A of T such that A is open and $x \in A$ and $A \subseteq$ it.

We introduce neighbourhood of x in T as a synonym of neighbourhood of x.

Next we state two propositions:

(24) Let T be a non empty topological space and A be a subset of T. Then A is open if and only if for every point x of T such that $x \in A$ there exists a neighbourhood B of x in T such that $B \subseteq A$.

(26)[3] Let T be a non empty topological space and A be a subset of T. Suppose that for every point x of T such that $x \in A$ holds A is a neighbourhood of x in T. Then A is open.

Let T be a topological structure. We say that T is T_1 if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let p, q be points of T. Suppose $p \neq q$. Then there exist subsets W, V of T such that W is open and V is open and $p \in W$ and $q \notin W$ and $q \in V$ and $p \notin V$.

Next we state the proposition

(27) For every non empty topological space T holds T is a T_1 space iff for every point p of T holds $\{p\}$ is closed.

Let T be a non empty topological space, let F be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let x be a point of T. Then $F(x)$ is a real number.

Next we state four propositions:

(28) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be open subsets of T. Suppose $A \neq \emptyset$ and $\overline{A} \subseteq B$. Then there exists a subset C of T such that $C \neq \emptyset$ and C is open and $\overline{A} \subseteq C$ and $\overline{C} \subseteq B$.

(29) Let T be a non empty topological space. Then T is a T_3 space if and only if for every open subset A of T and for every point p of T such that $p \in A$ there exists an open subset B of T such that $p \in B$ and $\overline{B} \subseteq A$.

(30) Let T be a non empty topological space. Suppose T is a T_4 space and a T_1 space. Let A be an open subset of T. If $A \neq \emptyset$, then there exists a subset B of T such that $B \neq \emptyset$ and $\overline{B} \subseteq A$.

(31) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose A is open and B is closed and $B \neq \emptyset$ and $B \subseteq A$. Then there exists a subset C of T such that C is open and $B \subseteq C$ and $\overline{C} \subseteq A$.

3. Some Increasing Family of Sets in Normal Space

Let T be a non empty topological space and let A, B be subsets of T. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is open and B is open and $\overline{A} \subseteq B$. A subset of T is called a between of A, B if:

(Def. 10) It $\neq \emptyset$ and it is open and $\overline{A} \subseteq$ it and $\overline{\text{it}} \subseteq B$.

One can prove the following proposition

[3] The proposition (25) has been removed.

(32) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let n be a natural number and G be a function from dyadic(n) into $2^{\text{the carrier of } T}$. Suppose $A \subseteq G(0)$ and $B = \Omega_T \setminus G(1)$ and for all elements r_1, r_2 of dyadic(n) such that $r_1 < r_2$ holds $G(r_1)$ is open and $G(r_2)$ is open and $\overline{G(r_1)} \subseteq G(r_2)$. Then there exists a function F from dyadic$(n+1)$ into $2^{\text{the carrier of } T}$ such that

(i) $A \subseteq F(0)$,

(ii) $B = \Omega_T \setminus F(1)$, and

(iii) for all elements r_1, r_2, r of dyadic$(n+1)$ such that $r_1 < r_2$ holds $F(r_1)$ is open and $F(r_2)$ is open and $\overline{F(r_1)} \subseteq F(r_2)$ and if $r \in$ dyadic(n), then $F(r) = G(r)$.

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