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Subalgebras of the Universal Algebra. Lattices of Subalgebras

Ewa Burakowska
Warsaw University
Białystok

Summary. Introduces a definition of a subalgebra of a universal algebra. A notion of similar algebras and basic operations on subalgebras such as a subalgebra generated by a set, the intersection and the sum of two subalgebras were introduced. Some basic facts concerning the above notions have been proved. The article also contains the definition of a lattice of subalgebras of a universal algebra.

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The articles [9], [5], [10], [11], [3], [1], [8], [4], [6], [12], [2], and [7] provide the notation and terminology for this paper.

One can prove the following propositions:

(1) For every natural number n and for every non empty set D and for every non empty subset D_1 of D holds $D^n \cap {D_1}^n = {D_1}^n$.

(2) For every non empty set D and for every homogeneous quasi total non empty partial function h from D^* to D holds $\mathrm{dom}\, h = D^{\mathrm{arity}\, h}$.

We follow the rules: U_0, U_1, U_2, U_3 are universal algebras, n is a natural number, and x is a set.

Let D be a non empty set. A non empty set is called a set of universal functions on D if:

(Def. 1) Every element of it is a homogeneous quasi total non empty partial function from D^* to D.

Let D be a non empty set and let P be a set of universal functions on D. We see that the element of P is a homogeneous quasi total non empty partial function from D^* to D.

Let us consider U_1. A set of universal functions on U_1 is a set of universal functions on the carrier of U_1.

Let U_1 be a universal algebra structure. A partial function on U_1 is a partial function from (the carrier of U_1)* to the carrier of U_1.

Let us consider U_1, U_2. We say that U_1 and U_2 are similar if and only if:

(Def. 2) $\mathrm{signature}\, U_1 = \mathrm{signature}\, U_2$.

Let us notice that the predicate U_1 and U_2 are similar is reflexive and symmetric.

Next we state three propositions:

(3) If U_1 and U_2 are similar, then $\mathrm{len}(\text{the characteristic of } U_1) = \mathrm{len}(\text{the characteristic of } U_2)$.

(4) If U_1 and U_2 are similar and U_2 and U_3 are similar, then U_1 and U_3 are similar.

(5) rng(the characteristic of U_0) is a non empty subset of (the carrier of U_0)$^* \dot{\rightarrow}$the carrier of U_0.

Let us consider U_0. The functor Operations(U_0) yielding a set of universal functions on U_0 is defined as follows:

(Def. 3) Operations(U_0) = rng(the characteristic of U_0).

Let us consider U_1. An operation of U_1 is an element of Operations(U_1).

In the sequel A is a non empty subset of U_0 and o is an operation of U_0.

We now state the proposition

(6) For every set n such that $n \in$ dom(the characteristic of U_0) holds (the characteristic of U_0)(n) is an operation of U_0.

Let U_0 be a universal algebra, let A be a subset of U_0, and let o be an operation of U_0. We say that A is closed on o if and only if:

(Def. 4) For every finite sequence s of elements of A such that $\operatorname{len} s = \operatorname{arity} o$ holds $o(s) \in A$.

Let U_0 be a universal algebra and let A be a subset of U_0. We say that A is operations closed if and only if:

(Def. 5) For every operation o of U_0 holds A is closed on o.

Let us consider U_0, A, o. Let us assume that A is closed on o. The functor o_A yielding a homogeneous quasi total non empty partial function from A^* to A is defined as follows:

(Def. 6) $o_A = o \restriction A^{\operatorname{arity} o}$.

Let us consider U_0, A. The functor Opers(U_0,A) yields a finite sequence of operational functions of A and is defined by the conditions (Def. 7).

(Def. 7)(i) dom Opers(U_0,A) = dom(the characteristic of U_0), and

(ii) for every set n and for every o such that $n \in$ dom Opers(U_0,A) and o = (the characteristic of U_0)(n) holds (Opers(U_0,A))$(n) = o_A$.

One can prove the following propositions:

(7) For every non empty subset B of U_0 such that B = the carrier of U_0 holds B is operations closed and for every o holds $o_B = o$.

(8) Let U_1 be a universal algebra, A be a non empty subset of U_1, and o be an operation of U_1. If A is closed on o, then arity(o_A) = arity o.

Let us consider U_0. A universal algebra is said to be a subalgebra of U_0 if it satisfies the conditions (Def. 8).

(Def. 8)(i) The carrier of it is a subset of U_0, and

(ii) for every non empty subset B of U_0 such that B = the carrier of it holds the characteristic of it = Opers(U_0,B) and B is operations closed.

Let U_0 be a universal algebra. One can verify that there exists a subalgebra of U_0 which is strict.

One can prove the following propositions:

(9) Let U_0, U_1 be universal algebras, o_0 be an operation of U_0, o_1 be an operation of U_1, and n be a natural number. Suppose that

(i) U_0 is a subalgebra of U_1,

(ii) $n \in$ dom(the characteristic of U_0),

(iii) o_0 = (the characteristic of U_0)(n), and

(iv) o_1 = (the characteristic of U_1)(n).

Then arity o_0 = arity o_1.

(10) If U_0 is a subalgebra of U_1, then dom(the characteristic of U_0) = dom(the characteristic of U_1).

(11) U_0 is a subalgebra of U_0.

(12) If U_0 is a subalgebra of U_1 and U_1 is a subalgebra of U_2, then U_0 is a subalgebra of U_2.

(13) If U_1 is a strict subalgebra of U_2 and U_2 is a strict subalgebra of U_1, then $U_1 = U_2$.

(14) For all subalgebras U_1, U_2 of U_0 such that the carrier of $U_1 \subseteq$ the carrier of U_2 holds U_1 is a subalgebra of U_2.

(15) For all strict subalgebras U_1, U_2 of U_0 such that the carrier of $U_1 =$ the carrier of U_2 holds $U_1 = U_2$.

(16) If U_1 is a subalgebra of U_2, then U_1 and U_2 are similar.

(17) For every non empty subset A of U_0 holds $\langle A, \mathrm{Opers}(U_0, A)\rangle$ is a strict universal algebra.

Let U_0 be a universal algebra and let A be a non empty subset of U_0. Let us assume that A is operations closed. The functor $\langle A, \mathrm{Ops}\rangle$ yielding a strict subalgebra of U_0 is defined as follows:

(Def. 9) $\langle A, \mathrm{Ops}\rangle = \langle A, \mathrm{Opers}(U_0, A)\rangle$.

Let us consider U_0 and let U_1, U_2 be subalgebras of U_0. Let us assume that the carrier of U_1 meets the carrier of U_2. The functor $U_1 \cap U_2$ yields a strict subalgebra of U_0 and is defined by the conditions (Def. 10).

(Def. 10)(i) The carrier of $U_1 \cap U_2 =$ (the carrier of U_1) $\cap$ (the carrier of U_2), and

(ii) for every non empty subset B of U_0 such that $B =$ the carrier of $U_1 \cap U_2$ holds the characteristic of $U_1 \cap U_2 = \mathrm{Opers}(U_0, B)$ and B is operations closed.

Let us consider U_0. The functor $\mathrm{Constants}(U_0)$ yielding a subset of U_0 is defined by:

(Def. 11) $\mathrm{Constants}(U_0) = \{a; a$ ranges over elements of U_0: $\bigvee_{o\,:\,\text{operation of } U_0}$ (arity $o = 0 \ \wedge\ a \in \mathrm{rng}\, o)\}$.

Let I_1 be a universal algebra. We say that I_1 has constants if and only if:

(Def. 12) There exists an operation o of I_1 such that arity $o = 0$.

One can check that there exists a universal algebra which is strict and has constants.
Let U_0 be a universal algebra with constants. Note that $\mathrm{Constants}(U_0)$ is non empty.
One can prove the following three propositions:

(18) For every universal algebra U_0 and for every subalgebra U_1 of U_0 holds $\mathrm{Constants}(U_0)$ is a subset of U_1.

(19) For every universal algebra U_0 with constants and for every subalgebra U_1 of U_0 holds $\mathrm{Constants}(U_0)$ is a non empty subset of U_1.

(20) Let U_0 be a universal algebra with constants and U_1, U_2 be subalgebras of U_0. Then the carrier of U_1 meets the carrier of U_2.

Let U_0 be a universal algebra and let A be a subset of U_0. Let us assume that $\mathrm{Constants}(U_0) \neq \emptyset$ or $A \neq \emptyset$. The functor $\mathrm{Gen}^{\mathrm{UA}}(A)$ yielding a strict subalgebra of U_0 is defined by the conditions (Def. 13).

(Def. 13)(i) $A \subseteq$ the carrier of $\mathrm{Gen}^{\mathrm{UA}}(A)$, and

(ii) for every subalgebra U_1 of U_0 such that $A \subseteq$ the carrier of U_1 holds $\mathrm{Gen}^{\mathrm{UA}}(A)$ is a subalgebra of U_1.

The following two propositions are true:

(21) For every strict universal algebra U_0 holds $\mathrm{Gen}^{\mathrm{UA}}(\Omega_{\text{the carrier of } U_0}) = U_0$.

(22) Let U_0 be a universal algebra, U_1 be a strict subalgebra of U_0, and B be a non empty subset of U_0. If $B =$ the carrier of U_1, then $\mathrm{Gen}^{\mathrm{UA}}(B) = U_1$.

Let U_0 be a universal algebra and let U_1, U_2 be subalgebras of U_0. The functor $U_1 \sqcup U_2$ yields a strict subalgebra of U_0 and is defined by:

(Def. 14) For every non empty subset A of U_0 such that $A = (\text{the carrier of } U_1) \cup (\text{the carrier of } U_2)$ holds $U_1 \sqcup U_2 = \mathrm{Gen}^{\mathrm{UA}}(A)$.

One can prove the following four propositions:

(23) Let U_0 be a universal algebra, U_1 be a subalgebra of U_0, and A, B be subsets of U_0. If $A \neq \emptyset$ or $\mathrm{Constants}(U_0) \neq \emptyset$ and if $B = A \cup$ the carrier of U_1, then $\mathrm{Gen}^{\mathrm{UA}}(A) \sqcup U_1 = \mathrm{Gen}^{\mathrm{UA}}(B)$.

(24) For every universal algebra U_0 and for all subalgebras U_1, U_2 of U_0 holds $U_1 \sqcup U_2 = U_2 \sqcup U_1$.

(25) For every universal algebra U_0 with constants and for all strict subalgebras U_1, U_2 of U_0 holds $U_1 \cap (U_1 \sqcup U_2) = U_1$.

(26) For every universal algebra U_0 with constants and for all strict subalgebras U_1, U_2 of U_0 holds $U_1 \cap U_2 \sqcup U_2 = U_2$.

Let U_0 be a universal algebra. The functor $\mathrm{Sub}(U_0)$ yields a set and is defined as follows:

(Def. 15) For every x holds $x \in \mathrm{Sub}(U_0)$ iff x is a strict subalgebra of U_0.

Let U_0 be a universal algebra. Note that $\mathrm{Sub}(U_0)$ is non empty.

Let U_0 be a universal algebra. The functor $\bigsqcup_{(U_0)}$ yielding a binary operation on $\mathrm{Sub}(U_0)$ is defined as follows:

(Def. 16) For all elements x, y of $\mathrm{Sub}(U_0)$ and for all strict subalgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $\bigsqcup_{(U_0)}(x, y) = U_1 \sqcup U_2$.

Let U_0 be a universal algebra. The functor $\bigsqcap_{(U_0)}$ yields a binary operation on $\mathrm{Sub}(U_0)$ and is defined as follows:

(Def. 17) For all elements x, y of $\mathrm{Sub}(U_0)$ and for all strict subalgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $\bigsqcap_{(U_0)}(x, y) = U_1 \cap U_2$.

The following propositions are true:

(27) $\bigsqcup_{(U_0)}$ is commutative.

(28) $\bigsqcup_{(U_0)}$ is associative.

(29) For every universal algebra U_0 with constants holds $\bigsqcap_{(U_0)}$ is commutative.

(30) For every universal algebra U_0 with constants holds $\bigsqcap_{(U_0)}$ is associative.

Let U_0 be a universal algebra with constants. The lattice of subalgebras of U_0 yielding a strict lattice is defined by:

(Def. 18) The lattice of subalgebras of $U_0 = \langle \mathrm{Sub}(U_0), \bigsqcup_{(U_0)}, \bigsqcap_{(U_0)} \rangle$.

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