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Subtrees[1]

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Summary. The concepts of root tree, the set of successors of a node in decorated tree and sets of subtrees are introduced.

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The articles [15], [12], [17], [16], [2], [18], [10], [11], [8], [14], [13], [4], [1], [3], [5], [6], [7], and [9] provide the notation and terminology for this paper.

1. Root Tree and Successors of Node in Decorated Tree

Let us mention that every tree which is finite is also finite-order.

Next we state three propositions:

(1) For every decorated tree t holds $t\restriction\varepsilon_{\mathbb{N}} = t$.

(2) For every tree t and for all finite sequences p, q of elements of $\mathbb{N}$ such that $p \frown q \in t$ holds $t\restriction(p \frown q) = t\restriction p\restriction q$.

(3) Let t be a decorated tree and p, q be finite sequences of elements of $\mathbb{N}$. If $p \frown q \in \operatorname{dom} t$, then $t\restriction(p \frown q) = t\restriction p\restriction q$.

Let I_1 be a decorated tree. We say that I_1 is root if and only if:

(Def. 1) $\operatorname{dom} I_1 =$ the elementary tree of 0.

One can check that every decorated tree which is root is also finite.

We now state three propositions:

(4) For every decorated tree t holds t is root iff $\emptyset \in \operatorname{Leaves}(\operatorname{dom} t)$.

(5) For every tree t and for every element p of t holds $t\restriction p =$ the elementary tree of 0 iff $p \in \operatorname{Leaves}(t)$.

(6) For every decorated tree t and for every node p of t holds $t\restriction p$ is root iff $p \in \operatorname{Leaves}(\operatorname{dom} t)$.

Let us observe that there exists a decorated tree which is root and there exists a decorated tree which is finite and non root.

Let x be a set. One can verify that the root tree of x is finite and root.

Let I_1 be a tree. We say that I_1 is finite-branching if and only if:

[1]This article has been worked out during the visit of the author in Nagano in Summer 1994.

(Def. 2) For every element x of I_1 holds $\operatorname{succ} x$ is finite.

Let us observe that every tree which is finite-order is also finite-branching.

Let us note that there exists a tree which is finite.

Let I_1 be a decorated tree. We say that I_1 is finite-order if and only if:

(Def. 3) $\operatorname{dom} I_1$ is finite-order.

We say that I_1 is finite-branching if and only if:

(Def. 4) $\operatorname{dom} I_1$ is finite-branching.

Let us mention that every decorated tree which is finite is also finite-order and every decorated tree which is finite-order is also finite-branching.

Let us observe that there exists a decorated tree which is finite.

Let t be a finite-order decorated tree. Note that $\operatorname{dom} t$ is finite-order.

Let t be a finite-branching decorated tree. Observe that $\operatorname{dom} t$ is finite-branching.

Let t be a finite-branching tree and let p be an element of t. Note that $\operatorname{succ} p$ is finite.

The scheme *FinOrdSet* deals with a unary functor $\mathcal{F}$ yielding a set and a finite set $\mathcal{A}$, and states that:

For every natural number n holds $\mathcal{F}(n) \in \mathcal{A}$ iff $n < \operatorname{card} \mathcal{A}$

provided the following requirements are met:

- For every set x such that $x \in \mathcal{A}$ there exists a natural number n such that $x = \mathcal{F}(n)$,
- For all natural numbers i, j such that $i < j$ and $\mathcal{F}(j) \in \mathcal{A}$ holds $\mathcal{F}(i) \in \mathcal{A}$, and
- For all natural numbers i, j such that $\mathcal{F}(i) = \mathcal{F}(j)$ holds $i = j$.

Let X be a set. One can verify that there exists a finite sequence of elements of X which is one-to-one and empty.

The following proposition is true

(7) Let t be a finite-branching tree, p be an element of t, and n be a natural number. Then $p \frown \langle n \rangle \in \operatorname{succ} p$ if and only if $n < \operatorname{card} \operatorname{succ} p$.

Let t be a finite-branching tree and let p be an element of t. The functor $\operatorname{Succ} p$ yields an one-to-one finite sequence of elements of t and is defined by:

(Def. 5) $\operatorname{len} \operatorname{Succ} p = \operatorname{card} \operatorname{succ} p$ and $\operatorname{rng} \operatorname{Succ} p = \operatorname{succ} p$ and for every natural number i such that $i < \operatorname{len} \operatorname{Succ} p$ holds $(\operatorname{Succ} p)(i+1) = p \frown \langle i \rangle$.

Let t be a finite-branching decorated tree and let p be a finite sequence. Let us assume that $p \in \operatorname{dom} t$. The functor $\operatorname{succ}(t, p)$ yielding a finite sequence is defined by:

(Def. 6) There exists an element q of $\operatorname{dom} t$ such that $q = p$ and $\operatorname{succ}(t, p) = t \cdot \operatorname{Succ} q$.

Next we state the proposition

(8) Let t be a finite-branching decorated tree. Then there exists a set x and there exists a decorated tree yielding finite sequence p such that $t = x\text{-tree}(p)$.

Let t be a finite decorated tree and let p be a node of t. Note that $t{\restriction}p$ is finite.

Next we state the proposition

(10)[1] For every finite tree t and for every element p of t such that $t = t{\restriction}p$ holds $p = \emptyset$.

Let D be a non empty set and let S be a non empty subset of $\operatorname{FinTrees}(D)$. Observe that every element of S is finite.

[1] The proposition (9) has been removed.

2. Set of Subtrees of Decorated Tree

Let t be a decorated tree. The functor $\mathrm{Subtrees}(t)$ yields a set and is defined as follows:

(Def. 7) $\mathrm{Subtrees}(t) = \{t\restriction p : p$ ranges over nodes of $t\}$.

Let t be a decorated tree. Observe that $\mathrm{Subtrees}(t)$ is constituted of decorated trees and non empty.

Let D be a non empty set and let t be a tree decorated with elements of D. Then $\mathrm{Subtrees}(t)$ is a non empty subset of $\mathrm{Trees}(D)$.

Let D be a non empty set and let t be a finite tree decorated with elements of D. Then $\mathrm{Subtrees}(t)$ is a non empty subset of $\mathrm{FinTrees}(D)$.

Let t be a finite decorated tree. One can check that every element of $\mathrm{Subtrees}(t)$ is finite.

In the sequel x is a set and t, t_1, t_2 are decorated trees.

Next we state four propositions:

(11) $x \in \mathrm{Subtrees}(t)$ iff there exists a node n of t such that $x = t\restriction n$.

(12) $t \in \mathrm{Subtrees}(t)$.

(13) If t_1 is finite and $\mathrm{Subtrees}(t_1) = \mathrm{Subtrees}(t_2)$, then $t_1 = t_2$.

(14) For every node n of t holds $\mathrm{Subtrees}(t\restriction n) \subseteq \mathrm{Subtrees}(t)$.

Let t be a decorated tree. The functor $\mathrm{FixedSubtrees}(t)$ yields a subset of $[\!:\mathrm{dom}\,t, \mathrm{Subtrees}(t):\!]$ and is defined as follows:

(Def. 8) $\mathrm{FixedSubtrees}(t) = \{\langle p, t\restriction p\rangle : p$ ranges over nodes of $t\}$.

Let t be a decorated tree. Observe that $\mathrm{FixedSubtrees}(t)$ is non empty.

Next we state three propositions:

(15) $x \in \mathrm{FixedSubtrees}(t)$ iff there exists a node n of t such that $x = \langle n, t\restriction n\rangle$.

(16) $\langle \emptyset, t\rangle \in \mathrm{FixedSubtrees}(t)$.

(17) If $\mathrm{FixedSubtrees}(t_1) = \mathrm{FixedSubtrees}(t_2)$, then $t_1 = t_2$.

Let t be a decorated tree and let C be a set. The functor C-$\mathrm{Subtrees}(t)$ yields a subset of $\mathrm{Subtrees}(t)$ and is defined by:

(Def. 9) C-$\mathrm{Subtrees}(t) = \{t\restriction p; p$ ranges over nodes of t: $p \notin \mathrm{Leaves}(\mathrm{dom}\,t) \vee t(p) \in C\}$.

In the sequel C is a set.

One can prove the following propositions:

(18) $x \in C$-$\mathrm{Subtrees}(t)$ iff there exists a node n of t such that $x = t\restriction n$ but $n \notin \mathrm{Leaves}(\mathrm{dom}\,t)$ or $t(n) \in C$.

(19) C-$\mathrm{Subtrees}(t)$ is empty iff t is root and $t(\emptyset) \notin C$.

Let t be a finite decorated tree and let C be a set. The functor C-$\mathrm{ImmediateSubtrees}(t)$ yielding a function from C-$\mathrm{Subtrees}(t)$ into $(\mathrm{Subtrees}(t))^*$ is defined by the condition (Def. 10).

(Def. 10) Let d be a decorated tree. Suppose $d \in C$-$\mathrm{Subtrees}(t)$. Let p be a finite sequence of elements of $\mathrm{Subtrees}(t)$. If $p = (C$-$\mathrm{ImmediateSubtrees}(t))(d)$, then $d = d(\emptyset)$-$\mathrm{tree}(p)$.

3. Set of Subtrees of Set of Decorated Tree

Let X be a constituted of decorated trees non empty set. The functor $\mathrm{Subtrees}(X)$ yields a set and is defined as follows:

(Def. 11) $\mathrm{Subtrees}(X) = \{t \upharpoonright p : t$ ranges over elements of X, p ranges over nodes of $t\}$.

Let X be a constituted of decorated trees non empty set. Note that $\mathrm{Subtrees}(X)$ is constituted of decorated trees and non empty.

Let D be a non empty set and let X be a non empty subset of $\mathrm{Trees}(D)$. Then $\mathrm{Subtrees}(X)$ is a non empty subset of $\mathrm{Trees}(D)$.

Let D be a non empty set and let X be a non empty subset of $\mathrm{FinTrees}(D)$. Then $\mathrm{Subtrees}(X)$ is a non empty subset of $\mathrm{FinTrees}(D)$.

In the sequel X, Y denote non empty constituted of decorated trees sets.

Next we state three propositions:

(20) $x \in \mathrm{Subtrees}(X)$ iff there exists an element t of X and there exists a node n of t such that $x = t \upharpoonright n$.

(21) If $t \in X$, then $t \in \mathrm{Subtrees}(X)$.

(22) If $X \subseteq Y$, then $\mathrm{Subtrees}(X) \subseteq \mathrm{Subtrees}(Y)$.

Let t be a decorated tree. Observe that $\{t\}$ is non empty and constituted of decorated trees.

We now state two propositions:

(23) $\mathrm{Subtrees}(\{t\}) = \mathrm{Subtrees}(t)$.

(24) $\mathrm{Subtrees}(X) = \bigcup\{\mathrm{Subtrees}(t) : t$ ranges over elements of $X\}$.

Let X be a constituted of decorated trees non empty set and let C be a set. The functor C-$\mathrm{Subtrees}(X)$ yielding a subset of $\mathrm{Subtrees}(X)$ is defined by:

(Def. 12) C-$\mathrm{Subtrees}(X) = \{t \upharpoonright p; t$ ranges over elements of X, p ranges over nodes of t: $p \notin \mathrm{Leaves}(\mathrm{dom}\, t) \lor t(p) \in C\}$.

One can prove the following four propositions:

(25) $x \in C$-$\mathrm{Subtrees}(X)$ iff there exists an element t of X and there exists a node n of t such that $x = t \upharpoonright n$ but $n \notin \mathrm{Leaves}(\mathrm{dom}\, t)$ or $t(n) \in C$.

(26) C-$\mathrm{Subtrees}(X)$ is empty iff for every element t of X holds t is root and $t(\emptyset) \notin C$.

(27) C-$\mathrm{Subtrees}(\{t\}) = C$-$\mathrm{Subtrees}(t)$.

(28) C-$\mathrm{Subtrees}(X) = \bigcup\{C$-$\mathrm{Subtrees}(t) : t$ ranges over elements of $X\}$.

Let X be a non empty constituted of decorated trees set. Let us assume that every element of X is finite. Let C be a set. The functor C-$\mathrm{ImmediateSubtrees}(X)$ yielding a function from C-$\mathrm{Subtrees}(X)$ into $(\mathrm{Subtrees}(X))^*$ is defined by the condition (Def. 13).

(Def. 13) Let d be a decorated tree. Suppose $d \in C$-$\mathrm{Subtrees}(X)$. Let p be a finite sequence of elements of $\mathrm{Subtrees}(X)$. If $p = (C$-$\mathrm{ImmediateSubtrees}(X))(d)$, then $d = d(\emptyset)$-$\mathrm{tree}(p)$.

Let t be a tree. Note that there exists an element of t which is empty.

Next we state four propositions:

(29) For every finite decorated tree t and for every element p of $\mathrm{dom}\, t$ holds $\mathrm{len}\, \mathrm{succ}(t, p) = \mathrm{len}\, \mathrm{Succ}\, p$ and $\mathrm{dom}\, \mathrm{succ}(t, p) = \mathrm{dom}\, \mathrm{Succ}\, p$.

(30) For every finite tree yielding finite sequence p and for every empty element n of $\widehat{p}$ holds $\mathrm{card}\, \mathrm{succ}\, n = \mathrm{len}\, p$.

(31) Let t be a finite decorated tree, x be a set, and p be a decorated tree yielding finite sequence. Suppose $t = x$-$\mathrm{tree}(p)$. Let n be an empty element of $\mathrm{dom}\, t$. Then $\mathrm{succ}(t, n) =$ the roots of p.

(32) For every finite decorated tree t and for every node p of t and for every node q of $t \upharpoonright p$ holds $\mathrm{succ}(t, p \frown q) = \mathrm{succ}(t \upharpoonright p, q)$.

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