JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Subsets of Topological Spaces[1]

Mirosław Wysocki
Warsaw University
Białystok

Agata Darmochwał
Warsaw University
Białystok

Summary. The article contains some theorems about open and closed sets. The following topological operations on sets are defined: closure, interior and frontier. The following notions are introduced: dense set, boundary set, nowheredense set and set being domain (closed domain and open domain), and some basic facts concerning them are proved.

MML Identifier: `TOPS_1`.

WWW: `http://mizar.org/JFM/Vol1/tops_1.html`

The articles [2], [3], and [1] provide the notation and terminology for this paper.

In this paper T_1 denotes a 1-sorted structure and K, Q denote subsets of T_1.

One can prove the following propositions:

(2)[1] $K \cup \Omega_{(T_1)} = \Omega_{(T_1)}$.

(8)[2] $(\Omega_{(T_1)})^{\rm c} = \emptyset_{(T_1)}$.

(20)[3] $K \subseteq Q$ iff K misses $Q^{\rm c}$.

(21) If $K^{\rm c} = Q^{\rm c}$, then $K = Q$.

For simplicity, we adopt the following rules: T_1 is a topological space, G_1 is a topological structure, x is a set, P, Q are subsets of T_1, K, L are subsets of T_1, R, S are subsets of G_1, and T, W are subsets of G_1.

Next we state the proposition

(22) $\emptyset_{(T_1)}$ is closed.

Let T be a topological space. Observe that $\emptyset_T$ is closed.

One can prove the following two propositions:

(26)[4] $\overline{\overline{T}} = \overline{T}$.

(27) $\overline{\Omega_{(G_1)}} = \Omega_{(G_1)}$.

Let T be a topological space and let P be a subset of T. One can verify that $\overline{P}$ is closed.

The following proposition is true

[1]Supported by RPBP.III-24.C1.
[1] The proposition (1) has been removed.
[2] The propositions (3)–(7) have been removed.
[3] The propositions (9)–(19) have been removed.
[4] The propositions (23)–(25) have been removed.

(29)[5] R is closed iff $R^{\rm c}$ is open.

Let T be a topological space and let R be a closed subset of T. Note that $R^{\rm c}$ is open.
One can prove the following proposition

(30) R is open iff $R^{\rm c}$ is closed.

Let T be a topological space. Observe that there exists a subset of T which is open.
Let T be a topological space and let R be an open subset of T. Observe that $R^{\rm c}$ is closed.
The following propositions are true:

(31) If S is closed and $T \subseteq S$, then $\overline{T} \subseteq S$.

(32) $\overline{K} \setminus \overline{L} \subseteq \overline{K \setminus L}$.

(34)[6] If R is closed and S is closed, then $\overline{R \cap S} = \overline{R} \cap \overline{S}$.

(35) If P is closed and Q is closed, then $P \cap Q$ is closed.

(36) If P is closed and Q is closed, then $P \cup Q$ is closed.

(37) If P is open and Q is open, then $P \cup Q$ is open.

(38) If P is open and Q is open, then $P \cap Q$ is open.

(39) Let G_1 be a non empty topological space, R be a subset of G_1, and p be a point of G_1. Then $p \in \overline{R}$ if and only if for every subset T of G_1 such that T is open and $p \in T$ holds R meets T.

(40) If Q is open, then $Q \cap \overline{K} \subseteq \overline{Q \cap K}$.

(41) If Q is open, then $\overline{Q \cap \overline{K}} = \overline{Q \cap K}$.

Let G_1 be a topological structure and let R be a subset of G_1. The functor ${\rm Int}\,R$ yields a subset of G_1 and is defined by:

(Def. 1) ${\rm Int}\,R = \overline{R^{\rm c}}^{\rm c}$.

We now state several propositions:

(43)[7] ${\rm Int}(\Omega_{(T_1)}) = \Omega_{(T_1)}$.

(44) ${\rm Int}\,T \subseteq T$.

(45) ${\rm Int}\,{\rm Int}\,T = {\rm Int}\,T$.

(46) ${\rm Int}\,K \cap {\rm Int}\,L = {\rm Int}(K \cap L)$.

(47) ${\rm Int}(\emptyset_{(G_1)}) = \emptyset_{(G_1)}$.

(48) If $T \subseteq W$, then ${\rm Int}\,T \subseteq {\rm Int}\,W$.

(49) ${\rm Int}\,T \cup {\rm Int}\,W \subseteq {\rm Int}(T \cup W)$.

(50) ${\rm Int}(K \setminus L) \subseteq {\rm Int}\,K \setminus {\rm Int}\,L$.

(51) ${\rm Int}\,K$ is open.

Let T be a topological space and let K be a subset of T. One can check that ${\rm Int}\,K$ is open.
The following two propositions are true:

(52) $\emptyset_{(T_1)}$ is open.

[5] The proposition (28) has been removed.
[6] The proposition (33) has been removed.
[7] The proposition (42) has been removed.

(53) $\Omega_{(T_1)}$ is open.

Let T be a topological space. Note that $\emptyset_T$ is open and Ω_T is open.

Let T be a topological space. Observe that there exists a subset of T which is open and closed.

Let T be a non empty topological space. Note that there exists a subset of T which is non empty, open, and closed.

We now state several propositions:

(54) $x \in \mathrm{Int}\, K$ iff there exists Q such that Q is open and $Q \subseteq K$ and $x \in Q$.

(55) If R is open, then $\mathrm{Int}\, R = R$ and if $\mathrm{Int}\, P = P$, then P is open.

(56) If S is open and $S \subseteq T$, then $S \subseteq \mathrm{Int}\, T$.

(57) P is open iff for every x holds $x \in P$ iff there exists Q such that Q is open and $Q \subseteq P$ and $x \in Q$.

(58) $\overline{\mathrm{Int}\, T} = \overline{\mathrm{Int}\, \overline{\mathrm{Int}\, T}}$.

(59) If R is open, then $\overline{\mathrm{Int}\, \overline{R}} = \overline{R}$.

Let G_1 be a topological structure and let R be a subset of G_1. The functor $\mathrm{Fr}\, R$ yielding a subset of G_1 is defined as follows:

(Def. 2) $\mathrm{Fr}\, R = \overline{R} \cap \overline{R^{\mathrm{c}}}$.

The following propositions are true:

(61)[8] Let G_1 be a non empty topological space, R be a subset of G_1, and p be a point of G_1. Then $p \in \mathrm{Fr}\, R$ if and only if for every subset S of G_1 such that S is open and $p \in S$ holds R meets S and R^{c} meets S.

(62) $\mathrm{Fr}\, T = \mathrm{Fr}(T^{\mathrm{c}})$.

(63) $\mathrm{Fr}\, T \subseteq \overline{T}$.

(64) $\mathrm{Fr}\, T = \overline{T^{\mathrm{c}}} \cap T \cup (\overline{T} \setminus T)$.

(65) $\overline{T} = T \cup \mathrm{Fr}\, T$.

(66) $\mathrm{Fr}(K \cap L) \subseteq \mathrm{Fr}\, K \cup \mathrm{Fr}\, L$.

(67) $\mathrm{Fr}(K \cup L) \subseteq \mathrm{Fr}\, K \cup \mathrm{Fr}\, L$.

(68) $\mathrm{Fr}\, \mathrm{Fr}\, T \subseteq \mathrm{Fr}\, T$.

(69) If R is closed, then $\mathrm{Fr}\, R \subseteq R$.

(70) $\mathrm{Fr}\, K \cup \mathrm{Fr}\, L = \mathrm{Fr}(K \cup L) \cup \mathrm{Fr}(K \cap L) \cup \mathrm{Fr}\, K \cap \mathrm{Fr}\, L$.

(71) $\mathrm{Fr}\, \mathrm{Int}\, T \subseteq \mathrm{Fr}\, T$.

(72) $\mathrm{Fr}\, \overline{T} \subseteq \mathrm{Fr}\, T$.

(73) $\mathrm{Int}\, T$ misses $\mathrm{Fr}\, T$.

(74) $\mathrm{Int}\, T = T \setminus \mathrm{Fr}\, T$.

(75) $\mathrm{Fr}\, \mathrm{Fr}\, \mathrm{Fr}\, K = \mathrm{Fr}\, \mathrm{Fr}\, K$.

(76) P is open iff $\mathrm{Fr}\, P = \overline{P} \setminus P$.

(77) P is closed iff $\mathrm{Fr}\, P = P \setminus \mathrm{Int}\, P$.

[8] The proposition (60) has been removed.

Let G_1 be a topological structure and let R be a subset of G_1. We say that R is dense if and only if:

(Def. 3) $\overline{R} = \Omega_{(G_1)}$.

Next we state four propositions:

(79)[9] If R is dense and $R \subseteq S$, then S is dense.

(80) P is dense iff for every Q such that $Q \neq \emptyset$ and Q is open holds P meets Q.

(81) If P is dense, then for every Q such that Q is open holds $\overline{Q} = \overline{Q \cap P}$.

(82) If P is dense and Q is dense and open, then $P \cap Q$ is dense.

Let G_1 be a topological structure and let R be a subset of G_1. We say that R is boundary if and only if:

(Def. 4) $R^{\rm c}$ is dense.

Next we state several propositions:

(84)[10] R is boundary iff $\operatorname{Int} R = \emptyset$.

(85) If P is boundary and Q is boundary and closed, then $P \cup Q$ is boundary.

(86) P is boundary iff for every Q such that $Q \subseteq P$ and Q is open holds $Q = \emptyset$.

(87) Suppose P is closed. Then P is boundary if and only if for every Q such that $Q \neq \emptyset$ and Q is open there exists a subset G of T_1 such that $G \subseteq Q$ and $G \neq \emptyset$ and G is open and P misses G.

(88) R is boundary iff $R \subseteq \operatorname{Fr} R$.

Let G_1 be a topological structure and let R be a subset of G_1. We say that R is nowhere dense if and only if:

(Def. 5) $\overline{R}$ is boundary.

We now state several propositions:

(90)[11] If P is nowhere dense and Q is nowhere dense, then $P \cup Q$ is nowhere dense.

(91) If R is nowhere dense, then $R^{\rm c}$ is dense.

(92) If R is nowhere dense, then R is boundary.

(93) If S is boundary and closed, then S is nowhere dense.

(94) If R is closed, then R is nowhere dense iff $R = \operatorname{Fr} R$.

(95) If P is open, then $\operatorname{Fr} P$ is nowhere dense.

(96) If P is closed, then $\operatorname{Fr} P$ is nowhere dense.

(97) If P is open and nowhere dense, then $P = \emptyset$.

Let G_1 be a topological structure and let R be a subset of G_1. We say that R is condensed if and only if:

(Def. 6) $\operatorname{Int} \overline{R} \subseteq R$ and $R \subseteq \overline{\operatorname{Int} R}$.

[9] The proposition (78) has been removed.
[10] The proposition (83) has been removed.
[11] The proposition (89) has been removed.

We say that R is closed condensed if and only if:

(Def. 7) $R = \overline{\mathrm{Int}\, R}$.

We say that R is open condensed if and only if:

(Def. 8) $R = \mathrm{Int}\, \overline{R}$.

We now state a number of propositions:

(101)[12] R is open condensed iff R^{c} is closed condensed.

(102) If R is closed condensed, then $\mathrm{Fr}\, \mathrm{Int}\, R = \mathrm{Fr}\, R$.

(103) If R is closed condensed, then $\mathrm{Fr}\, R \subseteq \overline{\mathrm{Int}\, R}$.

(104) If R is open condensed, then $\mathrm{Fr}\, R = \mathrm{Fr}\, \overline{R}$ and $\mathrm{Fr}\, \overline{R} = \overline{R} \setminus R$.

(105) If R is open and closed, then R is closed condensed iff R is open condensed.

(106)(i) If R is closed and condensed, then R is closed condensed, and

(ii) if P is closed condensed, then P is closed and condensed.

(107)(i) If R is open and condensed, then R is open condensed, and

(ii) if P is open condensed, then P is open and condensed.

(108) If P is closed condensed and Q is closed condensed, then $P \cup Q$ is closed condensed.

(109) If P is open condensed and Q is open condensed, then $P \cap Q$ is open condensed.

(110) If P is condensed, then $\mathrm{Int}\, \mathrm{Fr}\, P = \emptyset$.

(111) If R is condensed, then $\mathrm{Int}\, R$ is condensed and $\overline{R}$ is condensed.

REFERENCES

[1] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[2] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[3] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

Received April 28, 1989

Published January 2, 2004

---

[12] The propositions (98)–(100) have been removed.